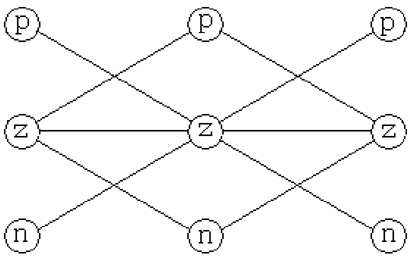
HW2 Solutions ECE 563 Fall 2009

1. a. $H(\chi) = \lim_{n\to\infty} H(X_n|X_1^{n-1}) = H(X_n|X_{n-1}) = H_b(\rho)$.
Or $H = -\sum_{i,j} \pi_j P_{ij} \log P_{ij} = H_b(\rho)$, since $\pi_1 = \pi_2 = \frac{1}{2}$
b. One possible Huffman code is

$$\begin{gathered}
a_0a_0a_0 \leftrightarrow 00000 \\
a_0a_0a_1 \leftrightarrow 0001 \\
a_0a_1a_0 \leftrightarrow 10 \\
a_0a_1a_1 \leftrightarrow 011 \\
a_1a_0a_0 \leftrightarrow 001 \\
a_1a_0a_1 \leftrightarrow 11 \\
a_1a_1a_0 \leftrightarrow 010 \\
a_1a_1a_1 \leftrightarrow 00001
\end{gathered}$$

c. $\bar{l} = \frac{\sum p_i l_i}{\text{num symbols}} = \frac{2.656}{3} = 0.885$
$\bar{l}^* = .874$. This huffman code is close to entropy, but could be made closer by using longer block lengths.

2.a. $B = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ so $\lambda = 2$ or $-1 \Rightarrow C = \log_2 \lambda_{max} = \log_2 2 = 1$ b. $B^4 = \begin{bmatrix} 6 & 5 \\ 10 & 11 \end{bmatrix}$
So you can find a set of 11 codewords of length 4.
c.



3.
$I(X;(Y,Z)) = H(Y,Z) - H(Y,Z|X) = H(Y) + H(Z|Y) - H(Y|X) - H(Z|Y,X)$
$= H(Y) - H(Y|X) + H(Z|Y) - H(Z|Y,X) = I(X;Y) + I(Z;X|Y)$

4. a. Assume X_1 and X_2 are independent. Then
$I(Y;X_1,X_2) = I(Y;X_1) + I(Y;X_2)$
$I(Y;X_1,X_2) = H(X_1,X_2) - H(X_1,X_2|Y)$
$= H(X_1) + H(X_2) - H(X_1|Y) - H(X_2|Y)$
$= [H(X_1) - H(X_1|Y)] + [H(X_2) - H(X_2|Y)]$
$= I(X_1;Y) + I(X_2;Y)$
b. Conditioning reduces entropy implies that $H(X) \geq H(X|Y)$, so
$I(X;Y) = H(X) - H(X|Y) \geq 0$.
c. $H(X|Y) = H(X)$ iff X and Y are independent, so equality in part b occurs iff X and Y are independent.

5. First consider when s is finite:

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & & & & 0 \\ 0 & 0 & 1 & 0 & \cdots & & & & 0 \\ \vdots & & \ddots & & & & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \ddots & & & \\ 1 & 0 & \cdots & & 0 & \cdots & 0 & 1 & 0 \\ 1 & 0 & \cdots & & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\lambda I - B = \begin{bmatrix} \lambda & -1 & 0 & 0 & \cdots & & & & 0 \\ 0 & \lambda & -1 & 0 & \cdots & & & & 0 \\ \vdots & & \ddots & & & & & & \\ 0 & 0 & \cdots & \lambda & -1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & \lambda & -1 & 0 & \cdots & 0 \\ \vdots & & & & & \ddots & & & \\ -1 & 0 & \cdots & & 0 & \cdots & \lambda & -1 & 0 \\ -1 & 0 & \cdots & & 0 & \cdots & 0 & \lambda & -1 \\ -1 & 0 & \cdots & & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$ So

$$\det(\lambda I - B) = \lambda^{s+1} + \det \begin{bmatrix} -1 & -1 & 0 & \cdots & & 0 \\ -1 & \lambda & -1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \\ -1 & 0 & \cdots & 0 & \lambda & -1 \\ -1 & 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}$$ So the characteristic polynomial is

$\det(\lambda I - B) = \lambda^{s+1} - \lambda^{s-r} - \lambda^{s-r+1} - \cdots - \lambda - 1 = 0.$

Which is the same as $\lambda^{s+1} - \frac{\lambda^{s-r+1}-1}{\lambda-1} = 0$, as desired.

If s is infinite, $B = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & & \\ 0 & 0 & 1 & 0 & \cdots & & \\ \vdots & & \ddots & & & & \\ 0 & 0 & \cdots & 1 & 0 & \cdots & \\ 1 & 0 & \cdots & 0 & 1 & \cdots & \\ \vdots & & & & & \ddots & \end{bmatrix}$ And the characteristic polynomial is $\det(\lambda I - B) = \lambda^r(\lambda - 1) - 1 = 0$. Since $\lambda = 1$ is not a root of the characteristic polynomial, the roots also satisfy $\left(\frac{x^{1-r}}{x-1}\right)\left(x^{r+1} - x^r - 1\right) = 0$

$x - \frac{x^{1-r}}{x-1} = 0.$