University of Illinois Fall 2006

ECE 413: Solutions to Problem Set 2

1. (a) $\binom{m+n}{k}$ committees of size k can be drawn from a set of m men and n women. Of these, $\binom{m}{i}\binom{n}{k-i}$ committees have i men and $k-i$ women. Summing over all possible values of i, we conclude that $\binom{m+n}{k} = \sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}$.

   (b) The coefficient of x^k in the polynomial $(1+x)^{m+n}$ is $\binom{m+n}{k}$. Now, if $h(x) = f(x)g(x)$, then $h_k = \sum_i f_i g_{k-i}$. Applying this to $f(x) = (1+x)^m$ and $g(x) = (1+x)^n$, we get that $\binom{m+n}{k} = \sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}$ as before.

   (c) With $m = k = n$, $\binom{2n}{n} = \sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{n}\left[\binom{n}{i}\right]^2$ since $\binom{n}{n-i} = \binom{n}{i}$.

   (d) $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+2)}{1\cdot 2\cdots(k-1)} \times \frac{n-k+1}{k} = \frac{n-k+1}{k}\binom{n}{k-1}$. Thus, we get that $\binom{n}{k} \geq \binom{n}{k-1}$ if $n-k+1 \geq k$, that is, if $k \leq (n+1)/2$. Thus, $\binom{n}{k}$ increases with k until $k > (n+1)/2$ from which point it decreases with increasing k. If $k = (n+1)/2$ (which can only occur when n is odd), there are twin peaks $\binom{n}{(n-1)/2} = \binom{n}{(n+1)/2}$. If n is even, then there is a unique maximum at $\binom{n}{n/2}$.

   (e)
$$\begin{aligned}(1+x)^n &= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \binom{n}{4}x^4 + \cdots\\ (1-x)^n &= \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 - \binom{n}{3}x^3 + \binom{n}{4}x^4 - \cdots\end{aligned}$$

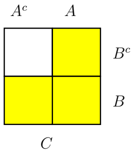
   from which we get that $(1+x)^n + (1-x)^n = 2\left[\binom{n}{0} + \binom{n}{2}x^2 + \binom{n}{4}x^4 + \cdots\right]$. Set $x = 1$ and note that the left side has value 2^n while the right side is twice the number of sets with an even number of elements. More simply, set $x = 1$ in $(1-x)^n$ to get

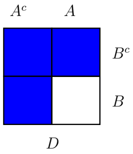
$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \Rightarrow \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots = 2^{n-1}.$$

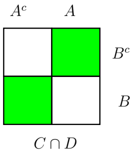
2. $\int_0^\infty\int_0^\infty \exp\left(-\frac{x^2+y^2}{2}\right)dx\,dy = \int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2}\exp\left(-\frac{r^2}{2}\right) r\,d\theta\,dr = \int_{\theta=0}^{\pi/2} 1\,d\theta = \frac{\pi}{2}.$

3. Since there are 31 *non-empty* subsets of the set of 5 flavors (you wouldn't want to buy unflavored ice-cream, would you?), the manufacturer can create and sell 31 specialty flavors.

4. The Karnaugh map below shows various events of interest.





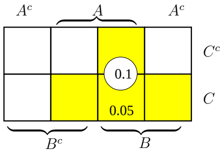


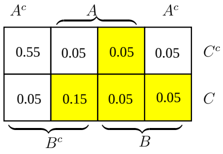
We are given that $P(C) = P(A \cup B) = 0.6$ and $P(D) = P(A^c \cup B^c) = 0.8$. Now, note that $C \cup D = \Omega$ while $C \cap D = (A \cup B) \cap (A^c \cup B^c) = (A \cap B^c) \cup (A^c \cap B) = A \oplus B$. Consequently, $P(C \cup D) = P(\Omega) = 1 = P(C) + P(D) - P(A \oplus B) = 0.6 + 0.8 - P(A \oplus B)$ from which we get that $P(A \oplus B) = 0.4$.

5. (a) Since there are $2^7 = 128$ outcomes with LSB = 1, we have that $P(A) = \dfrac{128}{256} = \dfrac{1}{2}$.
   (b) There are $\dbinom{8}{5} = \dbinom{8}{3} = 56$ outcomes in B and hence $P(B) = \dfrac{56}{256} = \dfrac{7}{32}$.
   (c) When the event $A \cap B$ occurs, we know that the 7 more significant bits have exactly 4 ONEs and 3 ZEROes. Hence $|A \cap B| = \dbinom{7}{4} = \dbinom{7}{3} = 35$. Thus, $P(A \cap B) = \dfrac{35}{256}$ and
   $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \dfrac{128}{256} + \dfrac{56}{256} - \dfrac{35}{256} = \dfrac{149}{256}$, while
   $P(A \oplus B) = P(A) + P(B) - 2 \cdot P(A \cap B) = \dfrac{128}{256} + \dfrac{56}{256} - 2 \cdot \dfrac{35}{256} = \dfrac{114}{256} = \dfrac{57}{128}$.
   (d) $P(C) = \dbinom{8}{4}/256 = \dfrac{70}{256} = \dfrac{35}{128}$.

6. (a) There are $\binom{6}{3} = 20$ possible short lists.
   (b) i. $P(\text{Beth is on the short list}) = \binom{5}{2}/20 = \frac{1}{2}$.
   ii. $P(\text{Chuck is on the short list}) = \binom{5}{2}/20 = \frac{1}{2}$ also.
   iii. $P(\text{Beth and Chuck both are on the short list}) = \binom{4}{1}/20 = \frac{1}{5}$.
   iv. $P(\text{Chuck and two women are on the short list}) = \binom{3}{2}/20 = \frac{3}{20}$.

7. (a) The Karnaugh map is as shown in the left hand figure below, with some probabilities marked on it. Note that the shaded region is the event $(A \cap B) \cup (B \cap C) \cup (A \cap C)$.





   (b) Since $A \cap B$ is the disjoint union of $A \cap B \cap C$ and $A \cap B \cap C^c$, we get that $P(A \cap B) = 0.1 = P(A \cap B \cap C) + P(A \cap B \cap C^c) = 0.05 + P(A \cap B \cap C^c)$ giving that $P(A \cap B \cap C^c) = 0.05$. Since $P(AB \cup BC \cup AC) = 0.3 = P(AB) + P(AB^cC) + P(A^cBC)$ while $P(AC) = P(ABC) + P(AB^cC) = 2P(BC) = P(ABC) + P(A^cBC)$, we readily obtain that $P(AB^cC) = 0.15$ and $P(A^cBC) = 0.05$. Since $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6$ we have that $P(A^cB^c) = 1 - P(A \cup B) = 0.4$. Since $P(A^cB^cC) = P(C) - P(BC) - P(AB^cC) = 0.05$, we get that $P(\text{cereal snaps, crackles, and pops}) = P(A^c \cap B^c \cap C^c) = 0.55$. See the figure on the right.
   (c) $P(\text{the sample fails } \textit{only} \text{ the snap test}) = P(AB^cC^c) = 0.05$.
   $P(\text{the sample fails } \textit{only} \text{ the crackle test}) = P(A^cBC^c) = 0.05$.
   $P(\text{the sample fails } \textit{only} \text{ the pop test}) = P(A^cB^cC) = 0.05$.