ECE 413: Solutions to Problem Set 3

1. (a) $\Gamma(1) = \int_0^\infty \exp(-x)\,dx = -\exp(-x)\Big|_0^\infty = 1.$

   (b) For $\alpha > 0$, $\Gamma(\alpha+1) = \int_0^\infty x^\alpha \exp(-x)\,dx = x^\alpha[-\exp(-x)]\Big|_0^\infty - \int_0^\infty \alpha x^{\alpha-1}[-\exp(-x)]\,dx$
   $= 0 + \alpha \int_0^\infty x^{\alpha-1}\exp(-x)\,dx = \alpha\Gamma(\alpha).$
   Hence $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)\cdots 2\cdot 1\cdot\Gamma(1) = n!$.

   (c) Similarly, $\Gamma(\alpha+1) = \alpha\Gamma(\alpha) = \alpha(\alpha-1)\Gamma(\alpha-1) = \cdots = \alpha(\alpha-1)(\alpha-2)\cdots\Gamma(\alpha-\lfloor\alpha\rfloor)$ where $\lfloor\alpha\rfloor$ is the *integer part* of α, and $0 < \alpha - \lfloor\alpha\rfloor < 1$.

   (d) Using the substitution $y = \sqrt{2x}$ and $dx/\sqrt{x} = \sqrt{2}dy$, we get

   $$\Gamma(1/2) = \int_0^\infty x^{-1/2}\exp(-x)\,dx = \sqrt{2}\int_0^\infty \exp(-y^2/2)\,dy = \sqrt{\pi}$$

   since from Problem 2 of Problem Set 2 we know that the value of the last integral is $\sqrt{\pi/2}$.

2. $P(A \cup (B^c \cup C^c)^c) = P(A \cup (B \cap C))$ by DeMorgan's theorem. Thus,

   (a) $B \cap C = \emptyset$ and hence $P(A \cup (B \cap C)) = P(A \cup \emptyset) = P(A) = 1/3$.

   (b) $P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap (B \cap C))$
   $= 4P(A \cap B \cap C) + 2P(A \cap B \cap C) - P(A \cap B \cap C) = 5P(A \cap B \cap C) = 5/8$.

   (c) $P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap (B \cap C)) = 1/2 + 1/3 - 0 = 5/6$. Why?

   (d) $(A \cup (B^c \cup C^c)^c)^c = A^c \cap (B^c \cup C^c)$. Hence, $P(A \cup (B^c \cup C^c)^c) = 1 - 0.6 = 0.4$.

3. (a)
   i. The sample space consists of 5-tuples of the form (B, C, B, B, C) where B and C have the obvious meaning, and 3 of the entrees must be B and 2 must be C. Obviously, the broccolous days can be chosen in $\binom{5}{3} = 10$ ways and hence $|\Omega| = 10$.
   ii. $P(\text{broccoli on Monday}) = \binom{4}{2}/\binom{5}{2} = 3/5$.
   iii. $P(\text{broccoli on Monday and Friday}) = \binom{3}{2}/\binom{5}{2} = 3/10$.
   iv. $P(\text{broccoli on Monday, Wednesday, and Friday}) = 1/\binom{5}{2} = 1/10$.

   (b) We now have 3 independent trials of an experiment.
   i. $P(\text{same veg on three days}) = P(AAA) + P(BBB) + P(CCC)$
   $= (0.2)^3 + (0.5)^3 + (0.3)^3 = 0.16$.
   ii. Let $A^c = B \cup C$.
   Then, $P(\text{same veg on two days}) = 3[P(AAA^c) + P(BBB^c) + P(CCC^c)]$
   $= 3[(0.2)^2 \times 0.8 + (0.5)^2 \times 0.5 + (0.3^2) \times 0.7] = 0.66$. Why the factor 3?
   iii. $P(\text{three different}) = 3!P(A)P(B)P(C) = 0.18$. But why 3!?? Alternatively, we can compute this as $1 - 0.16 - 0.66 = 0.18$. Why?

4. (a)
   i. $P(\Omega) = \sum_{n=0}^{\infty} \frac{(\ln 2)^n}{2(n!)} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(\ln 2)^n}{n!} = \frac{1}{2}\exp(\ln 2) = 1.$
   ii. $P(n \text{ is even}) = \sum_{n=0}^{\infty}\frac{(\ln 2)^{2n}}{2(2n)!}$. Now, $\exp(x) + \exp(-x) = 2\cosh(x) = 2\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$. Hence,
   $P(n \text{ is even}) = (1/4)[\exp(\ln 2) + \exp(-\ln 2)] = 5/8$.

(b) $P(B) = P(H) + P(TTTTH) + P(TTTTTTTTH) + \cdots = p + q^4p + q^8p + \cdots = \dfrac{p}{1-q^4}$.

$P(C) = P(TH) + P(TTTTTH) + P(TTTTTTTTTH) + \cdots = q[p + q^4p + q^8p + \cdots] = \dfrac{pq}{1-q^4}$.

Similarly, we have that $P(T) = \dfrac{pq^2}{1-q^4}$ and $P(A) = \dfrac{pq^3}{1-q^4}$. A nicer way of expressing these results is to note that $1 - q^4 = (1-q)(1+q+q^2+q^3)$ and therefore

$P(B) = \dfrac{1}{1+q+q^2+q^3}$, $P(C) = \dfrac{q}{1+q+q^2+q^3}$, $P(T) = \dfrac{q^2}{1+q+q^2+q^3}$,

$P(A) = \dfrac{q^3}{1+q+q^2+q^3}$.

Since $q < 1$, we have that $P(B) > P(C) > P(T) > P(A)$ which perhaps explains why Alice doesn't live here anymore. Also, quite obviously, $PB) + P(C) + P(T) + P(A) = 1$.