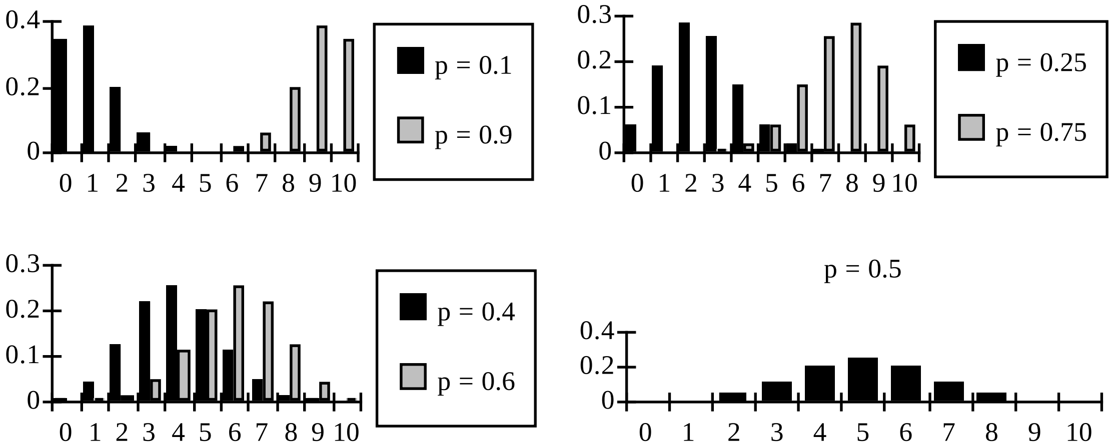
University of Illinois Fall 2006

ECE 413: Solutions to Problem Set 4

1. (a) (b) The graphs are as shown below.



   (c) The pmfs of $\mathcal{X}_p$ and $\mathcal{X}_{1-p}$ are "reverses" of each other. For each choice of k, $k = 0, 1, \ldots, 10$, and any value of p, $P\{\mathcal{X}_p = k\} = P\{\mathcal{X}_{1-p} = 10 - k\}$. Note that $\mathcal{X}_p$ counts the number of occurrences on 10 independent trials of an event A of probability p. But, if $\mathcal{X}_p = k$, then the *complementary event* A^c (of probability $1-p$), must have occurred $10 - k$ times, and the number of occurrences of A^c is counted by $\mathcal{X}_{1-p}$.

2. (a) $\mathcal{X}$ denotes a binomial random variable with parameters (N, p). It counts the number of occurrences of an event A of probability p on N independent trials. $\mathcal{Y} = N - \mathcal{X}$ counts the number of occurrences of A^c, an event of probability $1-p$, on the N independent trials and is thus a binomial random variable with parameters $(N, 1-p)$.

   (b) $P\{\mathcal{X} \text{ is even}\} = \binom{N}{0}(1-p)^N + \binom{N}{2}(1-p)^{N-2}p^2 + \binom{N}{4}(1-p)^{N-4}p^4 + \cdots$
   Now, a minor extension to Problem 1(e) of Problem Set 2 shows that $(x+y)^N + (x-y)^N = 2\left[\binom{N}{0}x^N + \binom{N}{2}x^{N-2}y^2 + \cdots\right]$, and so, setting $x = 1-p$, $y = p$, we get $P\{\mathcal{X} \text{ is even}\} = \frac{1}{2}\left[(1-p+p)^N + (1-p-p)^N\right] = \frac{1}{2}\left[1 + (1-2p)^N\right]$. Notice that the probability is $1/2$ for $p = 1/2$.

3. $\mathcal{X}$ takes on values $0, 1, 2, 3, 4, 5, 6, 7, 8$ with probabilities $p_{\mathcal{X}}(i)$ $\frac{1}{256}, \frac{8}{256}, \frac{28}{256}, \frac{56}{256}, \frac{70}{256}, \frac{56}{256}, \frac{28}{256}, \frac{8}{256}, \frac{1}{256}$.

   (a) $\mathsf{E}[\mathcal{X}] = \sum_{i=0}^{8} i \cdot p_{\mathcal{X}}(i) = 4$ by tedious calculation, or, more simply, $\mathsf{E}[\mathcal{X}] = np = 4$.

   (b) $\mathcal{Y}$ takes on values $1, 2, 3, 0$ with probabilities $\frac{28}{256}, \frac{8}{256}, \frac{1}{256}$, and $1 - \frac{28+8+1}{256} = \frac{219}{256}$.

   (c) i. $\mathsf{E}[\mathcal{Y}] = \sum_{i=0}^{3} i \cdot p_{\mathcal{Y}}(i) = \frac{47}{256}$. ii. $\mathsf{E}[\mathcal{Y}] = \sum_{i=0}^{8} \max\{\mathcal{X} - 5, 0\} \cdot p_{\mathcal{X}}(i) = \frac{47}{256}$

   (d) The airline has collected $8F$ in fares from the passengers. The operating cost is $200 + 10(\mathcal{X} - \mathcal{Y})$ and the airline pays out $\$(F+20)\mathcal{Y}$ in returned fares plus penalties. Thus, $\mathcal{Z} = 8F - 200 - 10(\mathcal{X} - \mathcal{Y}) - (F+20)\mathcal{Y} = 8F - 200 - 10\mathcal{X} - (F+10)\mathcal{Y}$ is the net profit from the flight. Hence, $\mathsf{E}[\mathcal{Z}] = 8F - 200 - 10\mathsf{E}[\mathcal{X}] - (F+10)\mathsf{E}[\mathcal{Y}] = 8F - 240 - (F+10)\frac{47}{256}$ is the average profit from this flight, which is 0 for $F \approx \$30.9395\ldots$. Note that on average, the airline keeps $7.81\ldots$ of the 8 fares that it collects and the average operating cost of the flight is only $\$238.16\ldots$. Including the penalties for denied boarding only increases the average cost to $\$241.83\ldots$.

(e) $\mathcal{Z}$ takes on values $8F-200$, $8F-210$, $8F-220$, $8F-230$, $8F-240$, $8F-250$, $7F-270$, $6F-290$, and $5F-310$ with probabilities $\frac{1}{256}, \frac{8}{256}, \frac{28}{256}, \frac{56}{256}, \frac{70}{256}, \frac{56}{256}, \frac{28}{256}, \frac{8}{256}, \frac{1}{256}$ respectively. For $F = 30.9395\ldots$, the values taken on by $\mathcal{Z}$ are $47.516\ldots$, $37.516\ldots$, $27.516\ldots$, $17.516\ldots$, $7.516\ldots$, $-2.48\ldots$, $-53.42\ldots$, $-104.36\ldots$ and $-155.30\ldots$. Thus, the airline loses money when more than 4 passengers show up but these losses occur with small probability, and the average profit is 0 (Verify this!)

4. Each die has probability $\frac{1}{6}$ of matching your chosen number. Thus, the number of dice that match is a binomial random variable with parameters $(3, \frac{1}{6})$.

   (a) (b) You lose \$6 if none of the dice show your chosen number. This occurs with probability $\left(\frac{5}{6}\right)^3 = \frac{125}{216}$. If one of the dice matches, you win \$6 with probability $3\cdot\frac{1}{6}\cdot\left(\frac{5}{6}\right)^2 = \frac{75}{216}$. You win \$12 with probability $3\cdot\frac{5}{6}\cdot\left(\frac{1}{6}\right)^2 = \frac{15}{216}$, and you win \$18 with probability $\left(\frac{1}{6}\right)^3 = \frac{1}{216}$. Rationality test: $125+75+15+1 = 216$.

   (b) $\mathsf{E}[\mathcal{X}] = (-6)\dfrac{125}{216} + 6\dfrac{75}{216} + 12\dfrac{15}{216} + 18\dfrac{1}{216} = -\dfrac{102}{216}$.

   (c) If all three dice show different numbers (which has probability $\frac{6\cdot5\cdot4}{6\cdot6\cdot6} = \frac{120}{216}$, you win \$1 on the three numbers but lose \$1 on the other three for a net gain of \$0. If two dice show the same number (which has probability $3\cdot\frac{6\cdot1\cdot5}{6\cdot6\cdot6} = \frac{90}{216}$ of occurring) you win \$2 for that number and \$1 for the other number showing, but lose \$1 on the remaining four for a net loss of \$1. If all three dice show the same number (which has probability $6\cdot\frac{1\cdot1\cdot1}{6\cdot6\cdot6} = \frac{6}{216}$ of occurring, you win \$3 for that number, but lose \$1 on the other five for a net loss of \$2. Thus,
   $$\mathsf{E}[\mathcal{Y}] = (0)\frac{120}{216} + (-1)\frac{90}{216} + (-2)\frac{6}{216} = -\frac{102}{216}.$$
   Thus, the average loss is the same regardless of whether you split your bet or not. Worse yet, your wealth will never increase if you split your bet: you will *never* win any money, whereas if you bet your money on one number, there is a small chance that you will be ahead at some time. (Do remember to quit while you are ahead, will ya?)

5. (a) $P\{\mathcal{Y} \text{ is even}\} = P\{\mathcal{Y}=0\} + P\{\mathcal{Y}=2\} + P\{\mathcal{Y}=4\} + \cdots = \exp(-\lambda)\left[1 + \dfrac{\lambda^2}{2!} + \dfrac{\lambda^4}{4!} + \cdots\right]$ $= \exp(-\lambda)\cosh(\lambda)$ by recognizing the series in square brackets.

   (b) $\dfrac{1}{2}\left[1 + (1-2p)^N\right] = \dfrac{1}{2}\left[1 + \left(1 - \dfrac{2\lambda}{N}\right)^N\right] \to \dfrac{1}{2}[1+\exp(-2\lambda)] = \exp(-\lambda)\dfrac{\exp(\lambda)+\exp(-\lambda)}{2}$ $= \exp(-\lambda)\cosh(\lambda)$.

   (c) The likelihood that we observed $\mathcal{Y} = k$ is $\exp(-\lambda)\lambda^k/k!$. This has derivative $[-\exp(-\lambda)\lambda^k + k\exp(-\lambda)\lambda^{k-1}]/k! = 0$ at $\lambda = k$. Thus, the maximum-likelihood estimate of λ when $\mathcal{Y} = k$ is observed is $\hat{\lambda} = k$. Note that for a binomial random variable, the maximum likelihoood estimate of p is k/n, i.e., the maximum-likelihood estimate of np is k, and recall that the Poisson random variable with parameter $\lambda = np$ approximates the binomial random variable. Therefore, our above result is consistent with the previous one.

6. (a) On average, $\mathsf{E}[\mathcal{X}] = 105\times0.9 = 94.5$ passengers show up for the flight.

   (b) $P\{\mathcal{X} \le 100\} = 1 - P\{\mathcal{X} > 100\}$
   $= 1 - P\{\mathcal{X}=101\} - P\{\mathcal{X}=102\} - P\{\mathcal{X}=103\} - P\{\mathcal{X}=104\} - P\{\mathcal{X}=105\}$
   $= 1 - \binom{105}{101}(0.9)^{101}(0.1)^4 - \binom{105}{102}(0.9)^{102}(0.1)^3 - \binom{105}{103}(0.9)^{103}(0.1)^3 - \binom{105}{104}(0.9)^{104}(0.1)^1 - \binom{105}{105}(0.9)^{105}(0.1)$
   $= 1 - \binom{105}{4}(0.9)^{101}(0.1)^4 - \binom{105}{3}(0.9)^{102}(0.1)^3 - \binom{105}{2}(0.9)^{103}(0.1)^3 - \binom{105}{1}(0.9)^{104}(0.1)^1 - \binom{105}{0}(0.9)^{105}(0.1) = 0.9832\ldots$

   (c) We saw earlier that if $\mathcal{X}$ is a binomial random variable with parameters (n,p), then $\mathcal{Y} = n - \mathcal{X}$ is a binomial random variable with parameters $(n, 1-p)$.

   (d) $P\{\mathcal{Y} \ge 5\} = 1 - P\{\mathcal{Y}=0\} - P\{\mathcal{Y}=1\} - P\{\mathcal{Y}=2\} - P\{\mathcal{Y}=3\} - P\{\mathcal{Y}=4\}$
   $= 1 - \exp(-10.5)\left[1 + \dfrac{10.5}{1!} + \dfrac{(10.5)^2}{2!} + \dfrac{(10.5)^3}{3!} + \dfrac{(10.5)^4}{4!}\right] = 0.9789\ldots$ which is close enough to $0.9832\ldots$ for gummint work.