University of Illinois Fall 2006

ECE 413: Solutions to Problem Set 5

1. (a) We are modeling the guesses as independent trials, and we are not allowing for other possibilities such as on some questions, the student can eliminate one or more alternatives and thus improve chances of getting the right answer to $\frac{1}{4}$ or $\frac{1}{3}$ etc.

   (b) The likelihood that we observed $\mathcal{W} = n$ is $\binom{N-K}{n}(0.8)^n(0.2)^{N-K-n}$ for $n = 0, 1, \ldots, N-K$.

   (c) For given N and n, the likelihood of part (b) is a function, say $f(K)$ of K. We have that

$$\frac{f(K)}{f(K-1)} = \frac{\binom{N-K}{n}(0.8)^n(0.2)^{N-K-n}}{\binom{N-(K-1)}{n}(0.8)^n(0.2)^{N-(K-1)-n}} = \frac{N-(K-1)-n}{0.2(N-(K-1))} \geq 1 \text{ iff } K \leq N - 1.25n + 1.$$

   Thus, the likelihood is maximum when K has value $\hat{K} = \lfloor N - 1.25n + 1 \rfloor$.

   (d) If n is *not* a multiple of 4, then the analysis of part (b) shows that $f(\hat{K})$ is the unique maximum of the likelihood, and the examiner's estimate $\tilde{K} = N - n - \lfloor 0.25n \rfloor$ equals the maximum likelihood estimate. When n *is* a multiple of 4, then $f(\hat{K}) = f(\hat{K}-1)$, and thus both $\hat{K}$ and $\hat{K}-1$ are legitimate maximum likelihood estimates of the unknown quantity. Note that in this case, the examiner's estimate $\tilde{K} = \hat{K} - 1$ and thus the examiner *is* using a maximum likelihood estimate (she is just a tough grader!).

   For $N = 100$ and $K = 90$, $\mathcal{W}$ can take on values $0, 1, \ldots, 10$. The value of $\mathcal{W}$ most likely to occur is 8. In this case, $\hat{K} = 91$ while $\tilde{K} = 90$ and so the examiner *does* estimate K correctly. On the other hand, if $\mathcal{W} = 4$, the examiner estimates K to be $\tilde{K} = 95$ (erring on the side of caution), and if $\mathcal{W} = 10$, then $\tilde{K} = 88$, ouch!

2. The number of red balls drawn is a binomial random variable $\mathcal{X}$ with parameters $(100, p)$ where $p = 10/(10 + x)$. We are told that $\mathcal{X} = 25$ on one experiment.

   (a) The maximum-likelihood estimate of x is denoted $\hat{x}$. Now, we can look at the ratio of the likelihoods of observing 25 red balls when the number of blue balls is k and $k - 1$ to get

$$\frac{\binom{100}{25}(10/(10+k))^{25}(k/(10+k))^{75}}{\binom{100}{25}(10/(9+k))^{25}((k-1)/(9+k))^{75}} = \left(\frac{9+k}{10+k}\right)^{100}\left(\frac{k}{k-1}\right)^{75}$$

   and try to figure out the k for which the ratio ≤ 1 which is messy. Alternatively, and more easily, we know that $\hat{p} = 0.25$ maximizes $\binom{100}{25}p^{25}(1-p)^{75}$. From this, we get that $\hat{p} = 0.25 = 10/(10 + \hat{x})$, and so $\hat{x} = 30$.

   (b) A confidence interval of length 0.2 gives a confidence level of $1 - 1/(100 \cdot 0.2^2) = 75\%$.

   (c) A confidence level of 96% results in a confidence interval of length $1/\sqrt{100 \cdot 0.04} = 0.5$. Note that the confidence interval is $(\hat{p} - 0.5, \hat{p} + 0.5) = (-0.25, 0.75)$ which can obviously be improved to $(0, 0.75)$ and further to $(0, 0.7142857\ldots)$ by arguing that x can take on integer values only, and so p cannot take on all real number values in the range $(0, 0.75)$.

3. (a) At least two boxes of Cornies must be bought.

   (b) For $k \geq 2$, $P\{\mathcal{X} = k\} = P(HH\cdots HB) + P(BB\cdots BH) = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{k-1} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{k-1}$.

$$\begin{aligned}
\mathsf{E}[\mathcal{X}] &= \sum_{k=2}^{\infty} k \cdot \left[\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{k-1} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{k-1}\right] \\
&= \sum_{k=1}^{\infty} k \cdot \left[\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{k-1} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{k-1}\right] - \frac{1}{3} - \frac{2}{3} = \frac{3}{1} + \frac{3}{2} - 1 = 3\frac{1}{2}.
\end{aligned}$$

It is instructive to do this problem via conditional pmfs. *Conditioned* on the first box being an L, the *conditional* pmf of $\mathcal{X}$ is that of $1+\mathcal{Y}$ where $\mathcal{Y}$ is a geometric random variable with parameter $\frac{1}{3}$ and mean 3. *Conditioned* on the first box being a D, the *conditional* pmf of $\mathcal{X}$ is that of $1+\mathcal{Z}$ where $\mathcal{Z}$ is a geometric random variable with parameter $\frac{2}{3}$ and mean 1.5. Thus, for $k \geq 2$,

$$\begin{aligned} P\{\mathcal{X}=k\} &= P\{\mathcal{X}=k \mid \text{first is } L\}P(L) + P\{\mathcal{X}=k \mid \text{first is } D\}P(D) \\ &= P\{\mathcal{Y}=k-1\}\left(\frac{2}{3}\right) + P\{\mathcal{Z}=k-1\}\left(\frac{1}{3}\right) \\ &= \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{k-2}\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{k-2}\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{k-1} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{k-1} \end{aligned}$$

as before. Similarly,

$$\mathsf{E}\mathcal{X} = \mathsf{E}\mathcal{X} \mid \text{first is } LP(L) + \mathsf{E}\mathcal{X} \mid \text{first is } DP(D) = (1+3)\left(\frac{2}{3}\right) + (1+1.5)\left(\frac{1}{3}\right) = 3\frac{1}{2}.$$

(c) Now, Mrs Kirk buys $\mathcal{W} \geq 4$ boxes of Cornies. Consider the contents of the first two boxes.

- If the first two boxes have given Jimmy one H and one K (this has probability $\frac{4}{9}$), then in effect he has been transported back in time to the previous year since he now just has to collect one H and one K. Thus, conditioned on the first two boxes having an H and a K, $\mathcal{W} = 2+\mathcal{X}$ where $\mathcal{X}$ was discussed in parts (a) and (b).
- If the first two boxes give Jimmy two H's (probability $\frac{4}{9}$) or two K's (probability $\frac{1}{9}$), then he has to wait for two K's (or two H's) to occur. Conditioned on this event HH (or KK), $\mathcal{W} = 2+\mathcal{V}$ where $\mathcal{V}$ is a *negative binomial* random variable with parameters $(2, \frac{1}{3})$ (or $(2, \frac{2}{3})$).

Hence, for $k \geq 4$, $P\{\mathcal{W}=k\} = P\{\mathcal{X}=k-2\}\frac{4}{9} + P\{\mathcal{V}=k-2\}\frac{4}{9} + P\{\mathcal{V}=k-2\}\frac{1}{9}$

$$= \left[\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{k-3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{k-3}\right]\frac{4}{9} + \left[(k-3)\left(\frac{1}{3}\right)^{2}\left(\frac{2}{3}\right)^{k-4}\right]\frac{4}{9} + \left[(k-3)\left(\frac{2}{3}\right)^{2}\left(\frac{1}{3}\right)^{k-4}\right]\frac{1}{9}$$

which simplifies to $P\{\mathcal{W}=k\} = \dfrac{(2^{k-2}+4)(k-1)}{3^k}$. Computing $\mathsf{E}[\mathcal{W}]$ from this result is messy. It is much easier to compute $\mathsf{E}[\mathcal{W}]$ from the conditional expectations $\mathsf{E}[\mathcal{W}|HH], \mathsf{E}[\mathcal{W}|KK]$ etc. as

$$\mathsf{E}[\mathcal{W}] = 2 + \left(3\frac{1}{2}\right)\frac{4}{9} + \left(\frac{2}{1/3}\right)\frac{4}{9} + \left(\frac{2}{2/3}\right)\frac{1}{9} = 6\frac{5}{9}.$$

4. (a) Assuming that the CRC detects all errors (which is not strictly true), the probability that the CRC indicates no error in a received packet is just $(1-p)^N$, the probability that all N bits were received correctly.

(b) $P\{\text{packet lost}\} = P\{\text{packet in error 5 times}\} = (1-(1-p)^N)^5$ and hence

$$\begin{aligned} &P\{\text{packet is received successfully}\} \\ &\quad = 1-(1-(1-p)^N)^5 \\ &\quad = 5(1-p)^N - 10(1-p)^{2N} + 10(1-p)^{3N} - 5(1-p)^{4N} + (1-p)^{5N}. \end{aligned}$$

(c) Let $Q = (1-p)^N$. Then, $P\{\mathcal{X}_i = 1\} = Q$. $P\{\mathcal{X}_i = 2\} = [1-Q]Q$. $P\{\mathcal{X}_i = 3\} = [1-Q]^2Q$. $P\{\mathcal{X}_i = 4\} = [1-Q]^3Q$. $P\{\mathcal{X}_i = 5\} = [1-Q]^4$. Note that the first four values are those corresponding to a geometric random variable with parameter Q, while the last is the probability that this random variable has value 5 or more..

$$\begin{aligned} \mathsf{E}[\mathcal{X}_i] &= Q + 2[1-Q]Q + 3[1-Q]^2Q + 4[1-Q]^3Q + 5[1-Q]^4 \\ &= 1 + [1-Q] + [1-Q]^2 + [1-Q]^3 + [1-Q]^4 \\ &= \frac{1-[1-Q]^5}{1-[1-Q]} = 5 - 10Q + 10Q^2 - 5Q^3 + Q^4. \end{aligned}$$

Verify that the sum telescopes from the right to the value shown.

(d) $P\{\text{none of the } L \text{ packets are lost}\} = (1-(1-(Q)^5)^L$.

5. (a) $P(A|B^c) = 1 - P(A^c|B^c) = 0.6$.
$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = 0.3 \times 0.7 + 0.6 \times 0.3 = 0.39$.
$P(B|A) = P(A|B)P(B)/P(A) = 0.3 \times 0.7/0.39 = 7/13$.

(b) $P(E \cap F) = P(F|E)P(E) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$ and $P(E|F) = \frac{P(E\cap F)}{P(F)} = \frac{1/8}{P(F)} = \frac{1}{3}$
$\Rightarrow\ P(F) = \frac{3}{8}$.

(c) $P(G \cup H) = P(G) + P(H) - P(G \cap H) = \frac{4}{3} - P(G \cap H) \leq 1$ from which we deduce that $P(G \cap H) \geq \frac{1}{3}$ and hence $P(G|H) = \frac{P(G\cap H)}{P(H)} \geq \frac{1/3}{2/3} = \frac{1}{2}$.

6. (a) Note that each row of the matrix (call it A) is just the *conditional* pmf of $\mathcal{Y}$ conditioned on the value of $\mathcal{X}$. Let $\vec{P}_{\mathcal{X}} = [0.5, 0.25, 0.25]$ be the *pmf vector* for $\mathcal{X}$. Then, $\vec{P}_{\mathcal{Y}} = \vec{P}_{\mathcal{X}}A$. More specifically, $P\{\mathcal{Y}=j\} = \sum_{i=1}^{3} P\{\mathcal{Y}=j|\mathcal{X}=i\}P\{\mathcal{X}=i\}$

$$= \begin{cases} 0.8 \times 0.5 + 0.05 \times 0.25 + 0.15 \times 0.25 & = 0.45 \quad \text{for } j = 1 \\ 0.1 \times 0.5 + 0.9 \times 0.25 + 0.05 \times 0.25 & = 0.2875 \quad \text{for } j = 2 \\ 0.1 \times 0.5 + 0.05 \times 0.25 + 0.8 \times 0.25 & = 0.2625 \quad \text{for } j = 3 \end{cases}$$

Sanity check: the three probabilities sum to 1.

(b) $P\{\mathcal{X}=i|\mathcal{Y}=3\} = \dfrac{P\{\mathcal{Y}=3|\mathcal{X}=i\}P\{\mathcal{X}=i\}}{P\{\mathcal{Y}=3\}} = \begin{cases} \frac{0.1\times0.5}{0.2625} & = \frac{4}{21} \quad \text{for } i=1, \\ \frac{0.05\times0.25}{0.2625} & = \frac{1}{21} \quad \text{for } i=2, \\ \frac{0.8\times0.25}{0.2625} & = \frac{16}{21} \quad \text{for } i=3. \end{cases}$
This is the conditional pmf of $\mathcal{X}$ given that $\mathcal{Y}$ was observed to have value 3. Note that the sum of the three probabilities is 1, as it should be for a valid pmf.

7. (a) This is sampling with replacement and we have that $P(R_1) = P(R_2) = \frac{r}{r+g}$.

(b) $P(R_2) = P(R_2|R_1)P(R_1) + P(R_2|R_1^c)P(R_1^c) = \frac{r+c}{r+c+g} \times \frac{r}{r+g} + \frac{r}{r+c+g} \times \frac{g}{r+g} = \frac{r}{r+g}$ just as before, *and independent of the value of* c!!

(c) $P\{r+c \text{ red balls}|R_2\} = \frac{P(R_1 \cap R_2)}{P(R_2)} = \frac{P(R_2|R_1)P(R_1)}{P(R_2)} = P(R_2|R_1) = \frac{r+c}{r+c+g}$

8. If the connecting flight is on time, everyone who shows up gets to board only if at most 85 of the 90 Chicagoans show up. If the connecting flight is late, all of the Chicagoans who show up can be accommodated.

(a) Hence, $P\{A|T\} = P\{\mathcal{Z} \leq 85\} = 1 - \sum_{k=86}^{90} \binom{90}{k} (0.9)^k (0.1)^{90-k} = 0.95345\ldots$ and $P\{A|T^c\} = 1$.

(b) $P\{\mathcal{X} \leq 100\} = P(A) = P\{A|T\}P(T) + P\{A|T^c\}P(T^c) = 0.95345 \times \frac{2}{3} + 1 \times \frac{1}{3} = 0.968966\ldots$.

(c) $P(T|\mathcal{X} \leq 100) = \frac{P\{\mathcal{X} \leq 100|T)P(T)}{P\{\mathcal{X} \leq 100\}} = \frac{0.95345 \times 2/3}{0.968966} = 0.65599\ldots$