ECE 413: Solutions to Problem Set 6

1. (a) Beth is on $\binom{5}{2} = 10$ of the 20 possible short lists, and hence $P(\text{Beth chosen}) = \frac{1}{2}$.

   (b) Chuck is also on 10 lists, but 4 of those 10 lists include Beth. Three of the other 6 lists include Di, and his conditional probability of being chosen is $\frac{1}{2}$. On the other 3 lists, his conditional probability of being chosen is $\frac{1}{3}$. Hence,
$$P(\text{Chuck chosen}) = 0 \times \frac{4}{20} + \frac{1}{2} \times \frac{3}{20} + \frac{1}{3} \times \frac{3}{20} = \frac{1}{8}.$$
   An alternative solution to this is instructive. Beth and Di are chosen with probabilities $\frac{1}{2}$ and 0 respectively. The other four must necessarily share the remaining probability equally because any argument that purports to show that Chuck has a better chance than Eddie (say) can be modified (by interchanging the two names everywhere) into an argument that shows that Eddie has a better chance than Chuck. Therefore, $P(\text{Chuck chosen}) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$.

   (c) Given that Beth was chosen as the monarch, the short list must have been one of the 10 with Beth on it. Four of these also had Di on it, giving $P(\text{Di on list} \mid \text{Beth chosen}) = \frac{4}{10} = \frac{2}{5}$. In fact, all the other five have equal chance $\frac{2}{5}$ of being on the short list with Beth. (Exercise: Explain why the probabilities add up to 2)

2. Since the pitcher can only pitch fast balls, curve balls, and sliders, we have that $P(F)+P(C)+P(S) = 1$, and since $P(C) = 2P(F)$, we conclude that $P(S) = 1 - 3P(F)$. We are also told that
$$\begin{aligned} P(H) = \frac{1}{4} &= P(H|F)P(F) + P(H|C)P(C) + P(H|S)P(S) \\ &= P(H|F)P(F) + P(H|C) \cdot 2P(F) + P(H|S)(1 - 3P(F)) \\ &= \frac{2}{5}P(F) + \frac{1}{4} \cdot 2P(F) + \frac{1}{6}(1 - 3P(F)) \end{aligned}$$
   from which we get that $P(F) = \dfrac{5}{24}, P(C) = \dfrac{10}{24}$, and $P(S) = \dfrac{9}{24}$.

3. Let R_1 denote the event that the ball drawn is red. Let H denote the event that the coin turned up Heads. Then, we are given that $P(R_1|H) = \frac{1}{2}$ and $P(R_1|H^c) = \frac{1}{3}$.

   (a) $P(R_1) = P(R_1|H)P(H) + P(R_1|H^c)P(H^c) = \dfrac{1}{2}p + \dfrac{1}{3}(1 - p) = \dfrac{1}{3} + \dfrac{p}{6} = \dfrac{2 + p}{6}$.
$$P(H^c|R_1) = \frac{P(R_1|H^c)P(H^c)}{P(R_1|H)P(H) + P(R_1|H^c)P(H^c)} = \frac{\frac{1}{3}(1-p)}{\frac{2+p}{6}} = \frac{2 - 2p}{2 + p}.$$

   (b) Let R_2 denote the event that the second ball drawn (without replacement) is red.
$$\begin{aligned} P(R_1R_2) &= P(R_1R_2|H)P(H) + P(R_1R_2|H^c)P(H^c) \\ &= \frac{1}{2}\left(\frac{x-1}{2x-1}\right)p + \frac{1}{3}\left(\frac{x-1}{3x-1}\right)(1-p) \\ &= \frac{(4x^2 - 6x + 2) + (5x^2 - 6x + 1)p}{(4x - 2)(9x - 3)}. \\ P(H^c|R_1R_2) &= \frac{P(R_1R_2|H^c)P(H^c)}{P(R_1R_2|H)P(H) + P(R_1R_2|H^c)P(H^c)} \\ &= \frac{\frac{(x-1)(1-p)}{9x-3}}{\frac{(4x^2-6x+2)+(5x^2-6x+1)p}{(4x-2)(9x-3)}} \\ &= \frac{(4x^2 - 6x + 2) - (4x^2 - 6x + 2)p}{(4x^2 - 6x + 2) + (5x^2 - 6x + 1)p}. \end{aligned}$$

4. Let A denote the event that your initial choice is the curtain concealing the car and B the event that your final choice is the curtain concealing the car. Clearly $P(A) = \frac{1}{3}$. Now the value of $P(B|A)$ and $P(B|A^c)$ (and therefore $P(B)$) depends on your *strategy* in response to Monty's blandishments.

   (a) Suppose that you always switch. Then, obviously $P(B|A) = 0$ since you chose curtain with the car initially and are now choosing the other curtain (with the goat.) Also, $P(B|A^c) = 1$ since you had a goat initially, and you know where the other goat is. Thus, $P(B) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{2}{3}$.

   (b) If you always stay put, then $P(B|A) = 1$ and $P(B|A^c) = 0$ leading to $P(B) = \frac{1}{3}$.

   (c) If, say, you toss a fair coin (independently!) to decide whether to stay put (H) or switch (T), then $P(B|A,H) = 1, P(B|A,T) = 0$ and therefore $P(B|A) = P(B|A,H)P(H) + P(B|A,T)P(T) = \frac{1}{2}$. Similarly $P(B|A^c) = \frac{1}{2}$. Monty is correct in his assertion. Besides, he said it on national TV! He wouldn't lie to you on national TV, would he?

   Many students are puzzled by the results of this problem. Consider that you have a $\frac{1}{3}$ chance of choosing the curtain with the car in the first place. The probability that one of the *other* two curtains is concealing the car is $\frac{2}{3}$. If you never switch, you win the car if you chose the right curtain to start with. Monty in essence is asking "Would you rather have what's behind both those two other two curtains except you don't *have* to take any goat(s) home with you unless you really want to, and oh, by the way, here is one goat behind this door, which is not telling you anything new since you knew that there was at least one goat behind the doors that you did not pick."

   Exercise: What would be your chances of winning if you tossed a biased coin?

   (d) It makes no difference whether you or your friend chooses first; you have equal probability of choosing the curtain with the car. He just happens to have been unlucky. In this game, but you should stay put because the chances are $\frac{2}{3}$ that the car is behind one of the two curtains picked by you and your friend. He's already gotten the goat, and so you get the car with probability $\frac{2}{3}$.

5. This game is different from the one in Problem 4 in that you have no idea what the rules of the game are. If the man is intent on separating you from your money as quickly as possible, he will not offer the chance to switch unless you picked the shell hiding the pea in the first place! That is, if you picked the wrong shell, the man will reveal the pea and you will lose your bet. Of course, if you look like a person willing to play several rounds, the man may set you up by playing by Monty's rules (and allowing you to win with probability 2/3) for some time. Then you will place a large bet on the wrong shell, and all of a sudden, you will not be given the choice of changing your bet! Personally, I would stick with the shell originally chosen since it gives me at least a 1/3 probability of winning regardless of the man's strategy; your experience (and monetary losses) may vary