University of Illinois Fall 2006

ECE 413: Solutions to Problem Set 7

1. (a) Yes, Fred and Wilma always alternate in tossing. Note that a game is won by the winner tossing a Head. The next toss starts a new game, and the loser goes first in the next game. Hence, Fred and Wilma always alternate in tossing. Alternation holds even for the other game considered on the exam in which the winner matches the previous toss. The winning toss of a game is considered to be the first toss of the next game. Since the loser of a game gets to toss the coin immediately after a winning toss (this toss by the loser of the previous game is the second toss of the next game), Fred and Wilma still alternate in tossing.

   (b) $P(F_{n+1}) = \frac{1}{3}P(F_n) + \frac{2}{3}P(W_n) = \frac{1}{3}P(F_n) + \frac{2}{3}(1 - P(F_n)) = \frac{2}{3} - \frac{1}{3}P(F_n)$.
   $P(W_{n+1}) = \frac{2}{3}P(F_n) + \frac{1}{3}P(W_n) = \frac{2}{3}(1 - P(W_n)) + \frac{1}{3}P(W_n) = \frac{2}{3} - \frac{1}{3}P(W_n)$.

   (c) Suppose that $P(F_n) = a + b\alpha^n$ for $n \geq 1$. Then we have that

   $$a + b\alpha^{n+1} = \frac{2}{3} - \frac{1}{3}\left(a + b\alpha^n\right) \Rightarrow b\alpha^n\left(\alpha + \frac{1}{3}\right) = \frac{2-4a}{3} \quad \text{for all } n \geq 1.$$

   In order for this to hold for all $n > 1$, it must be that $\alpha = -\frac{1}{3}$ and $a = \frac{1}{2}$. Hence,

   $$P(F_n) = \frac{1}{2} + b\left(-\frac{1}{3}\right)^n$$

   and since $P(F_1) = \frac{2}{3}$, we get that $b = -\frac{1}{2}$. The calculation for $P(W_n)$ is almost identical, except that the initial condition $P(W_1) = \frac{1}{3}$ gives a different value $+\frac{1}{2}$ for b. Hence, we get that

   $$P(F_n) = \frac{1}{2}\left[1 - \left(\frac{-1}{3}\right)^n\right] \quad \text{and} \quad P(W_n) = \frac{1}{2}\left[1 + \left(\frac{-1}{3}\right)^n\right].$$

   Note that $P(F_n) > P(W_n)$ for odd n, and $P(F_n) < P(W_n)$ for even n.

   (d) Since $\left(\frac{1}{3}\right)^n \to 0$ as $n \to \infty$, $\lim_{n\to\infty} P(F_n) = \lim_{n\to\infty} P(W_n) = \dfrac{1}{2}$ and the game is asymptotically fair.

2. (a) $p_0(86) = \binom{90}{86}(0.9)^{86}(0.1)^4$, $p_1(86) = \binom{90}{71}(0.9)^{71}(0.1)^{19}$. The likelihood ratio is $p_1(86)/p_0(86) \approx 0.027$ and hence the ML decision is that the connecting flight is late.

   (b) Repeat part (a) for the case when the gate agent observes that $\mathcal{X} = 96$. Obviously $p_0(96) = 0$, while $p_1(96) = \binom{90}{81}(0.9)^{81}(0.1)^9$. The likelihood ratio is $p_1(86)/p_0(86) = \infty$ and hence the ML decision is that the connecting flight is on time. In fact, it should be obvious that for $\mathcal{X} > 90$, the ML decision is that the connecting flight is on time.

   (c) The MAP decision rule is the same as the ML decision rule.

3. The likelihood matrix and joint probability matrices are as shown below, and the ML and MAP decision rules are indicated by boldface entries.

Hypothesis	$\mathcal{Y} = 1$	$\mathcal{Y} = 2$	$\mathcal{Y} = 3$	$P(H_i)$	$\mathcal{Y} = 1$	$\mathcal{Y} = 2$	$\mathcal{Y} = 3$
$\mathsf{H}_1 : \mathcal{X} = 1$	**0.8**	0.1	0.1	0.5	**0.4**	0.05	0.05
$\mathsf{H}_2 : \mathcal{X} = 2$	0.05	**0.9**	0.05	0.25	0.0125	**0.225**	0.0125
$\mathsf{H}_2 : \mathcal{X} = 3$	0.15	0.05	**0.8**	0.25	0.0375	0.0125	**0.2**

   Thus, the ML and MAP rules are the same in this case. Other *a priori* probabilities could make the MAP rule different from the ML rule . . .

4. (a) The likelihood of a hit is $P(H|F) = \frac{2}{5}$ or $P(H|C) = \frac{1}{4}$, or $P(H|S) = \frac{1}{6}$ depending on which hypothesis is true. Since $P(H|F) = \frac{2}{5}$ is the largest, the maximum-likelihood decision is that it was a fastball.

(b) We know from Problem Set 6 that $P(F) = \frac{5}{24}, P(C) = \frac{10}{24}$ and $P(S) = \frac{9}{24}$. Now we compare the joint probabilities

$$P(H|F)P(F) = \frac{2}{5} \times \frac{5}{24} = \frac{2}{24}, P(H|C)P(C) = \frac{1}{4} \times \frac{10}{24} = \frac{2.5}{24}, \text{and } P(H|S)P(S) = \frac{1}{6} \times \frac{9}{24} = \frac{1.5}{24}$$

to get the Bayesian decision that it was a curveball. Note that it is not necessary to find the *a posteriori* probabilities $P(F|H), P(C|H)$, and $P(S|H)$ explicitly; the joint probabilities suffice.

5. (a) The maximum-likelihood decision rule is indicated by shading in the likelihood matrix below.

Hypothesis	$\mathcal{X} = 3$	$\mathcal{X} = 6$	$\mathcal{X} = 9$	$\mathcal{X} = 12$
H_0: excellent	0.02	0.08	0.15	0.75
H_1: good	0.10	0.15	0.60	0.15
H_2: average	0.20	0.65	0.10	0.05

(b) P(excellent student is labeled as good) $= P(\mathcal{X} = 9|\mathsf{H}_0) = 0.15$.
P(excellent student is labeled as average) $= P(\{\mathcal{X} = 6\} \cup \{\mathcal{X} = 3\}|\mathsf{H}_0) = 0.02 + 0.08 = 0.1$.
P(average student is labeled as good or excellent) $= P(\{\mathcal{X} = 9\} \cup \{\mathcal{X} = 12\}|\mathsf{H}_2) = 0.15$.

(c) The conditional error probabilities of the maximum-likelihood decision rule are $P(E|\mathsf{H}_0) = 0.25$, $P(E|\mathsf{H}_1) = 0.4$, $P(E|\mathsf{H}_2) = 0.15$. Hence, the error probability is

$$P(E) = P(E|\mathsf{H}_0)\pi_0 + P(E|\mathsf{H}_1)\pi_1 + P(E|\mathsf{H}_2)\pi_2 = 0.05 + 0.22 + .0375 = 0.3075.$$

(d) The joint probability matrix is as shown below together with the MAP decision rule.

Hypothesis	$\mathcal{X} = 3$	$\mathcal{X} = 6$	$\mathcal{X} = 9$	$\mathcal{X} = 12$
H_0: excellent	0.0040	0.0160	0.0300	0.1500
H_1: good	0.0550	0.0825	0.3300	0.0825
H_2: average	0.0500	0.1625	0.0250	0.0125

$P(E) = 1 - (0.15 + 0.33 + 0.1625 + 0.055) = 0.3025$ which is slightly smaller than that of the maximum-likelihood rule. But note that students getting D's are classified as good while students getting C's are classified as average. Holy capricious grading complaint, Batman!

(e) Now the joint probability matrix looks as shown below, and all students are classified as excellent regardless of their grade on the exam!

Hypothesis	$\mathcal{X} = 3$	$\mathcal{X} = 6$	$\mathcal{X} = 9$	$\mathcal{X} = 12$
H_0: excellent	0.0190	0.0760	0.1425	0.7125
H_1: good	0.0050	0.0075	0.0003	0.0075
H_2: average	0.0000	0.000	0.0000	0.0000

Obviously, there is no need for examinations at the Lake Wobegon campus since the results are ignored anyway.

Noncredit exercise: Write a letter to the Governor asking him to demand that the University adopt the Lake Wobegon approach and eliminate all exams as a cost-cutting measure . . .