ECE 413: Solutions to Problem Set 8

1. **(a)** $P(\text{wins on first roll}) = \dfrac{6}{36} + \dfrac{2}{36} = \dfrac{2}{9}$. $P(\text{loses on first roll}) = \dfrac{1}{36} + \dfrac{2}{36} + \dfrac{1}{36} = \dfrac{1}{9}$.

$P(\text{establishes point } i \text{ on first roll}) = \dfrac{6 - |7-i|}{36}$, $i = 4, 5, 6, 8, 9, 10$. Check: probabilities sum to 1.

(b) If the shooter's point is i, then $P(\text{makes point} \mid \text{point is } i) = \dfrac{6-|7-i|}{36}$ while $P(\text{craps out}) = \dfrac{6}{36}$ so that $P(\text{game ends}) = \frac{12-|7-i|}{36}$. Else, the shooter rolls the dice again. Now, given disjoint events A and B, the probability that A occurs before B is the probability that a compound event of the form $(C, C, \ldots, C, A)$ occurs where $C = (A \cup B)^c$. Thus,

$$P(A \text{ occurs before } B) = P(A) + P(C)P(A) + [P(C)]^2 P(A) + \cdots = \frac{P(A)}{1-P(C)} = \frac{P(A)}{P(A)+P(B)}.$$

Hence, $P(\text{shooter wins} \mid \text{point is } i) = \dfrac{6-|7-i|}{12-|7-i|}$ which equals $\dfrac{3}{9}, \dfrac{4}{10}, \dfrac{5}{11}, \dfrac{5}{11}, \dfrac{4}{10}, \dfrac{3}{9}$ according as $4, 5, 6, 8, 9, 10$. The theorem of total probability now gives that

$$P(\text{shooter wins}) = \frac{2}{9} + 2\left[\frac{3}{9} \times \frac{3}{36} + \frac{4}{10} \times \frac{4}{36} + \frac{5}{11} \times \frac{5}{36}\right] = \frac{244}{495} = 0.492929\ldots$$

(c) Conditioned on the shooter's point being i, the number of *additional* dice rolls till the game ends is a geometric random variable with parameter $p = P(\text{game ends}) = \frac{12-|7-i|}{36}$ and conditional mean $\frac{36}{12-|7-i|}$. The expected number of dice rolls in a game of craps is thus

$$\mathsf{E}[\mathcal{X}] = \frac{3}{9} \times 1 + 2\left[\left(1 + \frac{36}{9}\right)\frac{3}{36} + \left(1 + \frac{36}{10}\right)\frac{4}{36} + \left(1 + \frac{36}{11}\right)\frac{5}{36}\right] = 3\frac{62}{165}.$$

(d) If the shooter's point is 8, then his chances of making the point the hard way are $\frac{1}{36}$ while the probability of making the point the easier way (or crapping out) are $\frac{6+4}{36}$. Thus, the shooter's probability of winning the hard way is $\frac{1}{1+10} = \frac{1}{11}$. Thus, the odds are perfectly fair. Ten times out of eleven, you will lose the dollar that you wager while one time out of eleven, you will win ten dollars.

2. **(a)** $F(u) = \begin{cases} 0 & u < 0, \\ u^2, & 0 \le u < 1, \\ 1, & u \ge 1. \end{cases}$ is a valid CDF. $P\{|\mathcal{X}| > 0.5\} = P\{\mathcal{X} > 0.5\} = 1 - F(0.5) = \frac{3}{4}$.

(b) $F(u) = \begin{cases} 0 & u < 1, \\ 2u - u^2, & 1 \le u \le 2, \\ 1, & u > 2. \end{cases}$ is *not* a valid CDF since $F(1) = 1 > F(2) = 0$.

(c) $F(u) = \begin{cases} \frac{1}{2}\exp(2u) & u \le 0, \\ 1 - \frac{1}{4}\exp(-3u), & u > 0, \end{cases}$ is *not* a valid CDF since it is not right-continuous at 0.

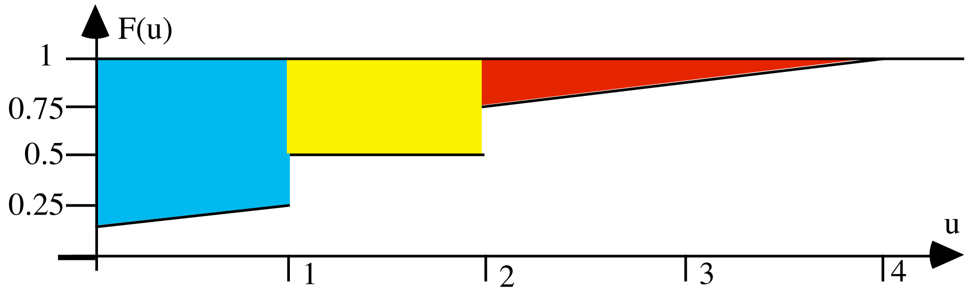
(d) $F(u) = \begin{cases} \frac{1}{2}\exp(2u) & u < 0, \\ 1 - \frac{1}{4}\exp(-3u), & u \ge 0, \end{cases}$ is a valid CDF.

$P\{|\mathcal{X}| > 0.5\} = 1 - P\{|\mathcal{X}| \le 0.5\} = 1 - (F(0.5) - F(-0.5)) = \frac{1}{2}\exp(-1) - \frac{1}{4}\exp(-1.5)$.

3. **(a)** $P\{\mathcal{X} = 2\} = F_{\mathcal{X}}(2^+) - F_{\mathcal{X}}(2^-) = \dfrac{1}{4}$, $P\{\mathcal{X} < 2\} = F_{\mathcal{X}}(2^-) = \dfrac{1}{2}$, $P\{\mathcal{X} > 2\} = 1 - F_{\mathcal{X}}(2) = \dfrac{1}{4}$,

$P\{1 \le \mathcal{X} \le 3\} = F_{\mathcal{X}}(3) - F_{\mathcal{X}}(1^-) = \dfrac{7}{8} - \dfrac{1}{4} = \dfrac{5}{8}$, and $P\{\mathcal{X} > 2 \mid \mathcal{X} > 0\} = \dfrac{P\{\mathcal{X} > 2\}}{P\{\mathcal{X} > 0\}} = \dfrac{2}{7}$.

(**b**) $\mathsf{E}[\mathcal{X}]$ is the area between the CDF and the line at height 1 as shown in the figure below. Elementary geometry gives $\mathsf{E}[\mathcal{X}] = \frac{1}{2} \times 1 \times \left(\frac{7}{8} + \frac{6}{8}\right) + \frac{1}{2} + \frac{1}{2} \times 2 \times \frac{1}{4} = \frac{22}{16} = 1.375.$



4. (**a**) We break the integral into a sum of integrals over the intervals $[k, k+1)$ to get

$$\mathsf{E}[\mathcal{X}] = \int_0^\infty P\{\mathcal{X} > u\}\,du = \sum_{k=0}^{\infty} \int_k^{k+1} P\{\mathcal{X} > u\}\,du.$$

But, for each $u \in [k, k+1)$, $P\{\mathcal{X} > u\}$ has the same value $P\{\mathcal{X} > k\}$. The k-th integral thus has a constant integrand $P\{\mathcal{X} > k\}$ and therefore evaluates to $P\{\mathcal{X} > k\}$. Furthermore, if $i = k+1$, then $P\{\mathcal{X} > k\} = P\{X \geq k+1\} = P\{X \geq i\}$. Hence $\mathsf{E}[\mathcal{X}] = \sum_{k=0}^{\infty} P\{X > k\} = \sum_{i=1}^{\infty} P\{X \geq i\}$.

(**b**) For $k = 0, 1, 2, \ldots$, $P\{\mathcal{X} > k\} = q^k$ where $q = 1 - p$. Therefore,

$$\mathsf{E}[\mathcal{X}] = \sum_{k=0}^{\infty} P\{X > k\} = \sum_{k=0}^{\infty} q^k = \frac{1}{1-q} = \frac{1}{p}.$$

(**c**) $\displaystyle\sum_{k=0}^{\infty} k \cdot P\{\mathcal{X} > k\} = \sum_{k=0}^{\infty} k \sum_{l=k+1}^{\infty} p_{\mathcal{X}}(l)$. Now, for any given l, the term $p_{\mathcal{X}}(l)$ will appear only in the sums for $k = 0, 1, \ldots, l-1$. Hence, (using LOTUS backwards!) we can write the sum as $\sum_{k=0}^{\infty} k \cdot P\{\mathcal{X} > k\} = \sum_{l=0}^{\infty} p_{\mathcal{X}}(l) \sum_{k=0}^{l-1} k = \sum_{l=0}^{\infty} p_{\mathcal{X}}(l) \cdot \frac{l(l-1)}{2} = \frac{1}{2}\mathsf{E}[\mathcal{X}(\mathcal{X}-1)]$.

5. If $f(u)$ is a nonnegative (or nonpositive) function with finite area A, then $A^{-1} \cdot f(u)$ is a valid pdf. This does not work if $f(u)$ takes on both positive and negative values.

(**a**) $f(u) = 2u, \quad 0 < u < 1$ is a valid pdf.
(**b**) $f(u) = |u|, \quad |u| < \frac{1}{2}$ is not a valid pdf, but $4 \cdot f(u)$ is.
(**c**) $f(u) = 1 - |u|, \quad |u| < 1$ is a valid pdf.
(**d**) $f(u) = \ln u, \quad 0 < u < 1$ is not a valid pdf but $-f(u)$ is.
(**e**) $f(u) = \ln u, \quad 0 < u < 2$ is not a valid pdf, nor is $C \cdot f(u)$ a valid pdf for any choice of C.
(**f**) $f(u) = \frac{2}{3}(u-1), \quad 0 < u < 3$, is not a valid pdf, nor is $C \cdot f(u)$ a valid pdf for any choice of C.
(**g**) $f(u) = \exp(-2u), \quad u > 0$ is not a valid pdf but $2 \cdot f(u)$ is.
(**h**) $f(u) = 4e^{-2u} - e^{-u}, \quad u > 0$, is not a valid pdf, nor is $C \cdot f(u)$ a valid pdf for any choice of C.
(**i**) $f(u) = \exp(-|u|), \quad |u| < 1$ is not a valid pdf but $e/[2(e-1)] \cdot f(u)$ is.

6. (**a**) $P\{\mathcal{X} > 20\} = \displaystyle\int_{20}^{\infty} \frac{10}{u^2}\,du = \left.\frac{-10}{u}\right|_{20}^{\infty} = \frac{1}{2}.$

(**b**) $F_{\mathcal{X}}(u_0) = \displaystyle\int_{-\infty}^{u_0} f_{\mathcal{X}}(u)\,du = \int_{10}^{u_0} \frac{10}{u^2}\,du = \left.\frac{-10}{u}\right|_{10}^{u_0} = 1 - \frac{10}{u_0}$. Note that $F_{\mathcal{X}}(20) = \frac{1}{2}$; it should be!

(**c**) $P\{\mathcal{X} > 15\} = 1 - F_{\mathcal{X}}(15) = \dfrac{10}{15} = \dfrac{2}{3}$. *Assuming that the device failures are independent*, the probability that at least 3 of the 6 last for 15 hours (i.e., at most 3 fail) is a binomial probability:

$$P\{\text{at least 3 of 6 devices work for 15 hours}\} = P\{\text{at most 3 failures}\} = \sum_{i=0}^{3} \binom{6}{i} \left(\frac{1}{3}\right)^i \left(\frac{2}{3}\right)^{6-i}.$$