University of Illinois Fall 2006

ECE 413: Solutions to Problem Set 9

1. (a) Given $\mathcal{X} = n$, it must be that $\mathcal{Y} + \mathcal{W} = n$. Therefore, whenever $k + l \neq n$, $P[\{\mathcal{Y} = k\} \cap \{\mathcal{W} = l\} \cap \{\mathcal{X} = n\}] = 0 \Rightarrow P[\{\mathcal{Y} = k\} \cap \{\mathcal{W} = l\}|\{\mathcal{X} = n\}] = 0$. When $n = k + l$,

$$P[\{\mathcal{Y} = k\} \cap \{\mathcal{W} = l\}|\{\mathcal{X} = k + l\}] = P[\{\mathcal{Y} = k\}|\{\mathcal{X} = k + l\}] = \binom{k+l}{k} p^k (1-p)^l.$$

(b) $\mathcal{Y}$ and $\mathcal{W}$ are *not* conditionally independent given the event $\{\mathcal{X} = n\}$. In fact, they are very much dependent since their sum is n.

(c) $P[\{\mathcal{Y} = k\} \cap \{\mathcal{W} = l\}] = \sum_{n=0}^{\infty} P[\{\mathcal{Y} = k\} \cap \{\mathcal{W} = l\}|\{\mathcal{X} = n\}]P\{\mathcal{X} = n\}$

$= \binom{k+l}{k} p^k (1-p)^l \exp(-\lambda) \frac{\lambda^{k+l}}{(k+l)!} = \exp(-\lambda p) \frac{(\lambda p)^k}{k!} \cdot \exp(-\lambda(1-p)) \frac{(\lambda(1-p))^l}{l!}$

$= P\{\mathcal{Y} = k\} \cdot P\{\mathcal{W} = l\}$.

(d) Yes, $\mathcal{Y}$ and $\mathcal{W}$ are *unconditionally* independent, even though they are conditionally very much dependent.

This particular result (called Poisson splitting) has many applications. For example, suppose that packets arriving at a server are placed into one of two queues with probabilities p and $(1-p)$ respectively. If the number of packets arriving per unit time is modeled as a Poisson random variable with parameter λ, then the numbers of packets in the queues can be modeled as independent Poisson random variables with parameters λp and $\lambda(1-p)$. Conversely, if the numbers of packets arriving per unit time on different ports are modeled as independent Poisson random variables with parameters λ_i, then the total number of packets is a Poisson random variable with parameter $\sum \lambda_i$.

2. (a) $F(u) = \begin{cases} 0 & u < 0, \\ u^2, & 0 \leq u < 1, \\ 1, & u \geq 1. \end{cases}$ is a valid CDF. $P\{|\mathcal{X}| > 0.5\} = P\{\mathcal{X} > 0.5\} = 1 - F(0.5) = \frac{3}{4}$.

(b) $F(u) = \begin{cases} 0 & u < 1, \\ 2u - u^2, & 1 \leq u \leq 2, \\ 1, & u > 2. \end{cases}$ is *not* a valid CDF since $F(1) = 1 > F(2) = 0$.

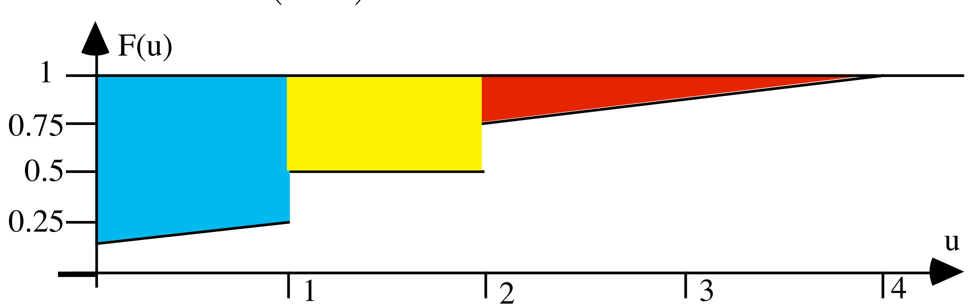
(c) $F(u) = \begin{cases} \frac{1}{2}\exp(2u) & u \leq 0, \\ 1 - \frac{1}{4}\exp(-3u), & u > 0, \end{cases}$ is *not* a valid CDF since it is not right-continuous at 0.

(d) $F(u) = \begin{cases} \frac{1}{2}\exp(2u) & u < 0, \\ 1 - \frac{1}{4}\exp(-3u), & u \geq 0, \end{cases}$ is a valid CDF.

$P\{|\mathcal{X}| > 0.5\} = 1 - P\{|\mathcal{X}| \leq 0.5\} = 1 - (F(0.5) - F(-0.5)) = \frac{1}{2}\exp(-1) - \frac{1}{4}\exp(-1.5)$.

3. (a) $P\{\mathcal{X} = 2\} = F_{\mathcal{X}}(2^+) - F_{\mathcal{X}}(2^-) = \frac{1}{4}$, $P\{\mathcal{X} < 2\} = F_{\mathcal{X}}(2^-) = \frac{1}{2}$, $P\{\mathcal{X} > 2\} = 1 - F_{\mathcal{X}}(2) = \frac{1}{4}$, $P\{1 \leq \mathcal{X} \leq 3\} = F_{\mathcal{X}}(3) - F_{\mathcal{X}}(1^-) = \frac{7}{8} - \frac{1}{4} = \frac{5}{8}$, and $P\{\mathcal{X} > 2 \mid \mathcal{X} > 0\} = \frac{P\{\mathcal{X} > 2\}}{P\{\mathcal{X} > 0\}} = \frac{2}{7}$.

(b) $\mathsf{E}[\mathcal{X}]$ is the area between the CDF and the line at height 1 as shown in the figure below. Elementary geometry gives $\mathsf{E}[\mathcal{X}] = \frac{1}{2} \times 1 \times \left(\frac{7}{8} + \frac{6}{8}\right) + \frac{1}{2} + \frac{1}{2} \times 2 \times \frac{1}{4} = \frac{22}{16} = 1.375$.



4. (a) We break the integral into a sum of integrals over the intervals $[k, k+1)$ to get

$$\mathsf{E}[\mathcal{X}] = \int_0^\infty P\{\mathcal{X} > u\}\, du = \sum_{k=0}^{\infty} \int_k^{k+1} P\{\mathcal{X} > u\}\, du.$$

But, for each $u \in [k, k+1)$, $P\{\mathcal{X} > u\}$ has the same value $P\{\mathcal{X} > k\}$. The k-th integral thus has a constant integrand $P\{\mathcal{X} > k\}$ and therefore evaluates to $P\{\mathcal{X} > k\}$. Furthermore, if $i = k+1$, then $P\{\mathcal{X} > k\} = P\{X \geq k+1\} = P\{X \geq i\}$. Hence $\mathsf{E}[\mathcal{X}] = \sum_{k=0}^{\infty} P\{X > k\} = \sum_{i=1}^{\infty} P\{X \geq i\}$.

(b) For $k = 0, 1, 2, \ldots$, $P\{\mathcal{X} > k\} = q^k$ where $q = 1 - p$. Therefore,

$$\mathsf{E}[\mathcal{X}] = \sum_{k=0}^{\infty} P\{X > k\} = \sum_{k=0}^{\infty} q^k = \frac{1}{1-q} = \frac{1}{p}.$$

(c) $\displaystyle\sum_{k=0}^{\infty} k \cdot P\{\mathcal{X} > k\} = \sum_{k=0}^{\infty} k \sum_{l=k+1}^{\infty} p_{\mathcal{X}}(l)$. Now, for any given l, the term $p_{\mathcal{X}}(l)$ will appear only in the sums for $k = 0, 1, \ldots, l-1$. Hence, (using LOTUS backwards!) we can write the sum as $\displaystyle\sum_{k=0}^{\infty} k \cdot P\{\mathcal{X} > k\} = \sum_{l=0}^{\infty} p_{\mathcal{X}}(l) \sum_{k=0}^{l-1} k = \sum_{l=0}^{\infty} p_{\mathcal{X}}(l) \cdot \frac{l(l-1)}{2} = \frac{1}{2}\mathsf{E}[\mathcal{X}(\mathcal{X}-1)]$.

5. If $f(u)$ is a nonnegative (or nonpositive) function with finite nonzero area A, then $A^{-1} \cdot f(u)$ is a valid pdf. This does not work if $f(u)$ takes on both positive and negative values.

(a) $f(u) = 2u, \quad 0 < u < 1$ is a valid pdf.

(b) $f(u) = |u|, \quad |u| < \frac{1}{2}$ is not a valid pdf, but $4 \cdot f(u)$ is.

(c) $f(u) = 1 - |u|, \quad |u| < 1$ is a valid pdf.

(d) $f(u) = \ln u, \quad 0 < u < 1$ is not a valid pdf but $-f(u)$ is.

(e) $f(u) = \ln u, \quad 0 < u < 2$ is not a valid pdf, nor is $C \cdot f(u)$ a valid pdf for any choice of C.

(f) $f(u) = \frac{2}{3}(u-1), \quad 0 < u < 3$, is not a valid pdf, nor is $C \cdot f(u)$ a valid pdf for any choice of C.

(g) $f(u) = \exp(-2u), \quad u > 0$ is not a valid pdf but $2 \cdot f(u)$ is.

(h) $f(u) = 4e^{-2u} - e^{-u}, \quad u > 0$, is not a valid pdf, nor is $C \cdot f(u)$ a valid pdf for any choice of C.

(i) $f(u) = \exp(-|u|), \quad |u| < 1$ is not a valid pdf but $e/[2(e-1)] \cdot f(u)$ is.

6. (a) $\displaystyle P\{\mathcal{X} > 20\} = \int_{20}^{\infty} \frac{10}{u^2}\, du = \left.\frac{-10}{u}\right|_{20}^{\infty} = \frac{1}{2}.$

(b) $\displaystyle F_{\mathcal{X}}(u_0) = \int_{-\infty}^{u_0} f_{\mathcal{X}}(u)\, du = \int_{10}^{u_0} \frac{10}{u^2}\, du = \left.\frac{-10}{u}\right|_{10}^{u_0} = 1 - \frac{10}{u_0}$. Note that $F_{\mathcal{X}}(20) = \frac{1}{2}$; it should be!

(c) $\displaystyle P\{\mathcal{X} > 15\} = 1 - F_{\mathcal{X}}(15) = \frac{10}{15} = \frac{2}{3}$. *Assuming that the device failures are independent*, the probability that at least 3 of the 6 last for 15 hours (i.e., at most 3 fail) is a binomial probability:

$$P\{\text{at least 3 of 6 devices work for 15 hours}\} = P\{\text{at most 3 failures}\} = \sum_{i=0}^{3} \binom{6}{i} \left(\frac{1}{3}\right)^i \left(\frac{2}{3}\right)^{6-i}.$$