ECE 413: Solutions to Problem Set 10

1. (a) Let A_n denote the event that the mailman is *not* bitten on the n-th day. Then, $\{\mathcal{X} = n\} = A_n^c \cap A_{n-1} \cap A_{n-2} \cap \cdots \cap A_1$ and $\{\mathcal{X} > n\} = A_n \cap A_{n-1} \cap A_{n-2} \cap \cdots \cap A_1$. We are given that $P\{A_n^c | A_{n-1} \cap A_{n-2} \cap \cdots \cap A_1\} = \dfrac{1}{n+1}$ and hence $P\{A_n | A_{n-1} \cap A_{n-2} \cap \cdots \cap A_1\} = \dfrac{n}{n+1}$. Note also that $P\{A_1\} = P\{A_1^c\} = \dfrac{1}{2}$. Therefore,

$$\begin{aligned} P\{\mathcal{X} > n\} &= P\{A_n \cap A_{n-1} \cap A_{n-2} \cap \cdots \cap A_1\} \\ &= P\{A_n | A_{n-1} \cap \cdots \cap A_1\} P\{A_{n-1} | A_{n-2} \cap \cdots \cap A_1\} \cdots P\{A_2 | A_1\} P\{A_1\} \\ &= \frac{n}{n+1} \times \frac{n-1}{n} \times \cdots \times \frac{2}{3} \times \frac{1}{2} = \frac{1}{n+1}. \end{aligned}$$

Hence $p_{\mathcal{X}}(n) = P\{\mathcal{X} = n\} = P\{\mathcal{X} > n-1\} - P\{\mathcal{X} > n\} = \dfrac{1}{n} - \dfrac{1}{n+1} = \dfrac{1}{n(n+1)}$, $n = 1, 2, \ldots$.

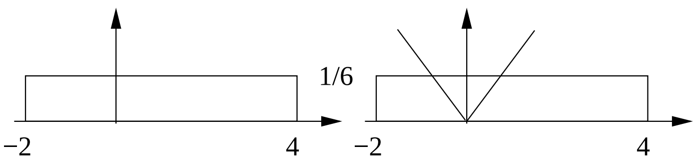
(b) $\mathsf{E}[\mathcal{X}] = \displaystyle\sum_{n=1}^{\infty} n \cdot P\{\mathcal{X} = n\} = \sum_{n=1}^{\infty} n \cdot \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty$ from the fact that the harmonic series diverges.

(c) $\mathsf{E}[\mathcal{X}] = \displaystyle\sum_{n=0}^{\infty} P\{\mathcal{X} > n\} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \infty$.

(d) Obviously, for $n = 1, 2, \ldots$, $P\{\mathcal{Y} = n\} = P\{\mathcal{Y} = -n\} = \dfrac{1}{2n(n+1)}$. Hence,

$\mathsf{E}[\mathcal{Y}] = \displaystyle\sum_{n=1}^{\infty} n \cdot P\{\mathcal{X} = n\} + \sum_{n=1}^{\infty} (-n) \cdot P\{\mathcal{X} = -n\} = \sum_{n=1}^{\infty} \frac{1}{n+1} - \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty - \infty$ and thus we see that $\mathsf{E}[\mathcal{Y}]$ is undefined.

2. A random variable uniformly distributed on $[a, b]$ has mean $\dfrac{a+b}{2}$ and variance $\dfrac{(b-a)^2}{12}$. Hence, we have that $b - a = 6$, and $b + a = 2$, giving $a = -2, b = 4$. The pdf is thus as shown below.



(a) By inspection, $P\{\mathcal{X} < 0\} = \frac{1}{3}$.

(b) $\mathsf{E}[|\mathcal{X}|] = \dfrac{1}{6}\left[\displaystyle\int_{-2}^{0} -u\,du + \int_{0}^{4} u\,du\right] = \dfrac{1}{6}\left[\left.\dfrac{-u^2}{2}\right|_{-2}^{0} + \left.\dfrac{u^2}{2}\right|_{0}^{4}\right] = \dfrac{1}{6}[2+8] = \dfrac{5}{3}$.

3. (a) It is easy to see that

$$\begin{aligned} \{Y > T\} &= (\{\mathcal{X}_1 > T\} \cap \{\mathcal{X}_2 > T\} \cap \{\mathcal{X}_3 > T\}) \cup (\{\mathcal{X}_1 > T\} \cap \{\mathcal{X}_2 > T\} \cap \{\mathcal{X}_3 \le T\}) \\ &\quad \cup (\{\mathcal{X}_1 > T\} \cap \{\mathcal{X}_2 \le T\} \cap \{\mathcal{X}_3 > T\}) \cup (\{\mathcal{X}_1 \le T\} \cap \{\mathcal{X}_2 > T\} \cap \{\mathcal{X}_3 > T\}) \end{aligned}$$

(b) Since $P\{\mathcal{X}_i > T\} = \exp(-\lambda T)$, we get that

$$P\{\mathcal{Y} > T\} = (\exp(-\lambda T))^3 + 3(\exp(-\lambda T))^2(1 - \exp(-\lambda T)) = 3\exp(-2\lambda T) - 2\exp(-3\lambda T).$$

Hence, $\mathsf{E}[\mathcal{Y}] = \displaystyle\int_0^{\infty} P\{\mathcal{Y} > T\}\,dT = \int_0^{\infty} 3\exp(-2\lambda T) - 2\exp(-3\lambda T)\,dT = \frac{3}{2\lambda} - \frac{2}{3\lambda} = \frac{5}{6}\lambda^{-1}$.

(c) Let $p = \exp(-\lambda T)$. Then, $P\{\mathcal{Y} > T\} = 3p^2 - 2p^3 = \frac{1}{2}$, which has only one solution $p = 1/2$ in the range $0 \leq p \leq 1$. Thus, the median value of $\mathcal{Y}$ is the solution T to $\exp(-\lambda T) = 1/2$, giving $T = \lambda^{-1}\ln 2$.

(d) The MTBF λ^{-1} of the single module is larger (!) than the MTBF of the TMR system. The median lifetimes are the same. The TMR system *does not* result in a more reliable system if MTBF and median lifetimes are the criteria.

(e) If $\lambda = -\ln 0.999$, then $\exp(-\lambda)) = 0.999$. We have that $P\{\mathcal{X}_1 > 1\} = \exp(-\lambda)) = 0.999$ while $P\{\mathcal{Y} > 1\} = 3(0.999)^2 - 2(0.999)^3 = 0.999997002$ so that the TMR system is likely to provide much more reliability for one unit of time.

(f) The largest value of T for which $P\{\mathcal{Y} > T\} \geq 0.999$ is $-\lambda^{-1}\ln 0.98163 \approx 18.53$. Thus, the TMR system can be expected to work (with 99.9% reliability) for more than 18 units of time whereas the single module provides this level of reliability for just one unit of time. It is in matters such as these that the TMR systems shines . . . not in terms of comparisons of MTBF or median lifetimes.

4. (a) The derivative of $\exp(-u^2/2)$ is $-u\exp(-u^2/2)$. Hence,

$$\mathsf{E}[|\mathcal{X}|] = 2\int_0^\infty u \cdot \frac{1}{\sqrt{2\pi}}\exp(-u^2/2) = \sqrt{\frac{2}{\pi}}\left[-\exp(-u^2/2)\right|_0^\infty = \sqrt{\frac{2}{\pi}}.$$

More generally, if $\mathcal{X} \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathsf{E}[|\mathcal{X} - \mu|] = \sigma\sqrt{\frac{2}{\pi}}$ and is known as the *absolute error*.

(b) Since $t + x > t - x > 0$, we have that $(t + x)(t - x) = t^2 - x^2 > (t - x)^2 > 0$. Hence,

$$\exp(x^2/2)Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2 - x^2}{2}\right)dt \leq \int_x^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(t-x)^2}{2}\right)dt = \frac{1}{2}.$$

5. (a) $P\{\mathcal{X} < 0\} = \Phi\left(\frac{0 - (-10)}{2}\right) = \Phi(5) = 1 - Q(5)$.

(b) $P\{-10 < \mathcal{X} < 5\} = \Phi\left(\frac{5-(-10)}{2}\right) - \Phi\left(\frac{-10-(-10)}{2}\right) = \Phi(7.5) - \Phi(0) = \Phi(7.5) - \frac{1}{2} = \frac{1}{2} - Q(7.5)$.

(c) $P\{\,|\mathcal{X}| \geq 5\} = P\{\,\mathcal{X} \leq -5\} + P\{\,\mathcal{X} \geq 5\} = \Phi\left(\frac{-5-(-10)}{2}\right) + 1 - \Phi\left(\frac{5-(-10)}{2}\right) = \Phi(2.5) + 1 - \Phi(7.5) = 1 - Q(2.5) + Q(7.5)$.

(d) $P\{\mathcal{X}^2 - 3\mathcal{X} + 2 > 0\} = P\{(\mathcal{X} - 1)(\mathcal{X} - 2) > 0\} = P\{\mathcal{X} < 1\} + P\{\mathcal{X} > 2\} = \Phi\left(\frac{1-(-10)}{2}\right) + 1 - \Phi\left(\frac{2-(-10)}{2}\right) = \Phi(5.5) + 1 - \Phi(6) = 1 - Q(5.5) + Q(6)$.

6. Let $\mathcal{X} \sim \mathsf{N}(0.9, 0.003^2)$ denote the width (in microns) of the trace.

(a) $\{\mathcal{X} < 0.9 - 0.005\}$ or $\{\mathcal{X} > 0.9 + 0.005\}$ for a trace to be deemed defective.
$P\{\,|\mathcal{X} - 0.9| > 0.005\} = 2\Phi(-0.005/0.003) = 2\Phi(-1.666\ldots) = 2Q(1.666\ldots) \approx 0.095$.

(b) We need to find the maximum value of σ such that $2Q(0.005/\sigma) \leq 10^{-2}$. Since $Q(2.575) \approx 0.005$, we get that $0.0005/\sigma > 2.575$, that is, $\sigma \leq 0.005/2.575 \approx 0.00194$.