University of Illinois Fall 2006

ECE 413: Solutions to Problem Set 11

1. (a) $P\{\text{demand can be satisfied}\} = P\left\{\mathcal{X} \leq \frac{1}{2}\right\} = \int_0^{0.5} 5(1-u)^4\,du = -(1-u)^5\Big|_0^{0.5} = \frac{31}{32}.$

   (b) We want the minimum value of C such that $P\{\mathcal{X} > C\} \leq 10^{-5}$. But,
   $$P\{\mathcal{X} > C\} = \int_C^1 5(1-u)^4\,du = -(1-u)^5\Big|_C^1 = (1-C)^5 \leq 10^{-5} \text{ if } C \geq 0.9.$$

   (c) $\mathcal{Y} = \min\{\mathcal{X}, C\} = \begin{cases} \mathcal{X}, & \text{if } \mathcal{X} \leq C, \\ C, & \text{if } \mathcal{X} > C. \end{cases}$.

   (d) The weekly *gross* profit is \$640$\mathcal{Y}$ and the *average* weekly gross profit is
   $$\mathsf{E}[640\mathcal{Y}] = \int_0^C 5(1-u)^4(640u)\,du + \int_C^1 5(1-u)^4(640C)\,du = \frac{640}{6}\left[1-(1-C)^6\right].$$

   As a function of C, this increases from 0 at $C = 0$ to 640/6 when $C = 1$. The average *net* profit is \$$\mathsf{E}[(640\mathcal{Y} - 20C)] = \mathsf{E}[(640\mathcal{Y}] - 20C = \frac{640}{6}\left[1-(1-C)^6\right] - 20C$ whose maximum can be found by the usual calculus method to be at $C = \frac{1}{2}$. Thus, a 500 gallon tank that costs \$10 per week to rent gives an average gross profit of \$630/6 and the maximum average net profit of \$570/6 per week. In contrast, a full-sized 1000 gallon tank gives only a slightly larger gross profit of \$640/6 per week, but a smaller net profit of \$520/6 per week. Of course, there may be intangible losses in goodwill if the tank is emptied before the week ends. However, the probability of this happening is only $\frac{1}{32}$ when $C = \frac{1}{2}$.

2. (a) $\mathcal{I}$ can take on values in the range $(-I_0, \infty)$.

   (b) $F_{\mathcal{I}}(v) = 0$ for $v < -I_0$. For any $v > -I_0$,
   $F_{\mathcal{I}}(v) = P\{\mathcal{I} \leq v\} = P\{I_0(\exp(\mathcal{V}) - 1) \leq v\} = P\{\mathcal{V} \leq \ln(1 + v/I_0)\} = F_{\mathcal{V}}(\ln(1 + v/I_0))$.

   (c) For $v > -I_0$,
   $$f_{\mathcal{I}}(v) = f_{\mathcal{V}}(\ln(1+v/I_0))\frac{1}{1+v/I_0} \times \frac{1}{I_0} = \frac{f_{\mathcal{V}}(\ln(1+v/I_0))}{v+I_0} = \begin{cases} \frac{I_0/2}{(v+I_0)^2}, & v \geq 0, \\ \frac{1}{2I_0}, & -I_0 < v < 0, \end{cases}.$$
   Note that the pdf has constant value $1/(2I_0)$ from $v = -I_0$ to $v = 0$.

3. (a) $\mathcal{Y}$ takes on values in $[0,1]$ and hence $F_{\mathcal{Y}} = 0$ for $v < 0$, and $F_{\mathcal{Y}}(v) = 1$ for $v > 1$.
   For $0 \leq v \leq 1$, $F_{\mathcal{Y}}(v) = P\{\mathcal{Y} \leq v\} = P\{\mathcal{X}^2 \leq v\} = P\{-\sqrt{v} \leq \mathcal{X} \leq \sqrt{v}\} = \sqrt{v}$.
   Hence $f_{\mathcal{Y}}(v) = \frac{1}{2\sqrt{v}}$ if $0 \leq v \leq 1$, and $f_{\mathcal{Y}}(v) = 0$, otherwise.

   (b) $\mathcal{Z}$ takes on values in $[-1,1]$ and hence, $F_{\mathcal{Z}} = 0$ for $v < -1$, and $F_{\mathcal{Z}}(v) = 1$ for $v > 1$.
   For $0 \leq v \leq 1$, $F_{\mathcal{Z}}(v) = P\{\mathcal{Z} \leq v\} = P\{g(\mathcal{X}) \leq v\} = P\{\mathcal{X} \leq \sqrt{v}\} = \frac{1}{2}[1 + \sqrt{v}]$.
   For $-1 \leq v \leq 0$, $F_{\mathcal{Z}}(v) = P\{\mathcal{Z} \leq v\} = P\{g(\mathcal{X}) \leq v\} = P\{\mathcal{X} \leq \sqrt{-v}\} = \frac{1}{2}[1 - \sqrt{-v}]$.
   Hence, $f_{\mathcal{Z}}(v) = \frac{1}{4\sqrt{|v|}}$ if $0 \leq |v| \leq 1$, and $f_{\mathcal{Z}}(v) = 0$, otherwise. Note that the pdf is an even function, and approaches $+\infty$ as v approaches 0 from either side.

4. (a) The pmf of $\mathcal{Y}$ is $p_{\mathcal{Y}}(\alpha) = p_{\mathcal{Y}}(-\alpha) = \dfrac{1}{2}$.

   (b) $\mathsf{E}[\mathcal{Z}] = \int_0^\infty (u-\alpha)^2\phi(u)\,du + \int_{-\infty}^0 (u+\alpha)^2\phi(u)\,du = \int_{-\infty}^\infty (u^2+\alpha^2)\phi(u)\,du - 4\int_0^\infty \alpha u\phi(u)\,du$
   $= 1 + \alpha^2 - 2\sqrt{\frac{2}{\pi}}\alpha$ where we have used the properties of the standard Gaussian pdf $\phi(u)$ and Problem 4(a) of Problem Set 10 in arriving at the result. Now, a quadratic function $ax^2 + 2bx + c$ has extremum $c - \dfrac{b^2}{a}$ at $x = -\dfrac{b}{a}$, the average of the roots, and this extremum is a minimum if $a > 0$, maximum if $a < 0$. So we see $\mathsf{E}[\mathcal{Z}]$ has minimum value $1 - \dfrac{2}{\pi}$ at $\alpha = \sqrt{\frac{2}{\pi}}$.

(c) $p_{\mathcal{W}}(3) = p_{\mathcal{W}}(-3) = \Phi(-2.5) = 0.0062.$ $p_{\mathcal{W}}(2) = p_{\mathcal{W}}(-2) = \Phi(2.5) - \Phi(1.5) = 0.0606.$
$p_{\mathcal{W}}(1) = p_{\mathcal{W}}(-1) = \Phi(1.5) - \Phi(0.5) = 0.2417.$ $p_{\mathcal{W}}(0) = \Phi(0.5) - \Phi(-0.5) = 0.3830.$

(d) $\mathcal{Z}_2, \mathcal{Z}_1, \mathcal{Z}_0$ are *Bernoulli* random variables with parameters $p_2 = P\{\mathcal{W} < 0\} = 0.3085$, $p_1 = P\{\mathcal{W} \in \{-2,-1,2,3\}\} = 0.3691$, and $p_0 = P\{\mathcal{W} \in \{-3,-1,1,3\}\} = 0.4958$ respectively.

5. As discussed in class, the first arrival time $\mathcal{X}$ is an exponential random variable with parameter λ, and its pdf is $\lambda \exp(-\lambda t)$ for $t > 0$.

(a) Hence, $P\{\mathcal{X} \le T\} = \displaystyle\int_0^T \lambda \exp(-\lambda t)\,dt = 1 - \exp(-\lambda T) = \frac{\lambda T}{1!} - \frac{(\lambda T)^2}{2!} + \cdots$

(b) $P(A) = P\{N(0,T] = 1\} = \dfrac{\lambda T}{1!}\exp(-\lambda T) = \lambda T \exp(-\lambda T).$

(c) $P(A)$ in part (b) is *not* the same as $P\{\mathcal{X} \le T\}$ from part (a). In fact, $A \subset \{\mathcal{X} \le T\}$ since the latter event includes the possibility that the *second* arrival also occurred during $(0,T]$.

(d) For $0 < t < T$,

$$\begin{aligned}
P(\{X \le t\} \mid A) &= \frac{P(\{X \le t\} \cap A)}{P(A)} = \frac{P(\{N(0,t] = 1\} \cap \{N(t,T] = 0\})}{P\{N(0,T] = 1\}} \\
&= \frac{P\{N(0,t] = 1\} \times P\{N(t,T] = 0\}}{P\{N(0,T] = 1\}} \text{ by independence of disjoint intervals} \\
&= \frac{(\lambda t)\exp(-\lambda t) \times \exp(-\lambda(T-t))}{\lambda T \exp(-\lambda T)} = \frac{t}{T}.
\end{aligned}$$

Thus, the *conditional* pdf of the first arrival time $\mathcal{X}$ given that there was exactly one arrival in $(0,T]$ is a *uniform* pdf on $(0,T]$!!

6. I'm considering, I'm considering!

(a) The number of arrivals in $(0,4]$ is $N(0,4]$, a Poisson random variable with parameter 4λ. Hence, $\mathsf{E}[N(0,4]] = 4\lambda$.

(b) The event $\{N(0,3] = 3\} \cap \{N(2,6] = 0\}$ is exactly the same as $\{N(0,2] = 3\} \cap \{N(2,6] = 0\}$ and thus $P[\{N(0,3] = 3\} \cap \{N(2,6] = 0\}] = P[\{N(0,2] = 3\} \cap \{N(2,6] = 0\}]$. We get

$$P[\{N(0,3] = 3\} \cap \{N(2,6] = 0\}] = \frac{(2\lambda)^3}{3!}\exp(-2\lambda) \times \exp(-4\lambda) = \frac{4\lambda^3}{3}\exp(-6\lambda)$$

by independence of Poisson variables on disjoint intervals of time.

(c) The number of arrivals in $(0,6]$ is $N(0,6]$, a Poisson random variable with parameter 6λ. If it is observed that $N(0,6] = k$, then the maximum-likelihood estimate of the Poisson parameter 6λ is k (cf. Problem 3(c) of Problem Set 4), and hence the maximum-likelihood estimate of the arrival rate is $k/6 = \#$ of arrivals/length of interval which is exactly what one would expect.

(d) $P\{N(0,t) \ge 1\} = 1 - P\{N(0,t) = 0\} = 1 - \exp(-\lambda t) = 1 - 2^{-t}$ when $\lambda = \ln 2$.