University of Illinois Fall 2006

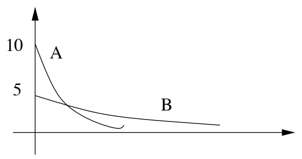
ECE 413: Solutions to Problem Set 12

1. $Y(f) = H(f)X(f) = \mathrm{rect}(f/2)\exp(-\pi f^2)$, and $y(0) = \int_{-\infty}^{\infty} Y(f)\,df = \int_{-1}^{1} \exp(-\pi f^2)\,df$. But, the *integrand* is the Gaussian pdf $\mathcal{N}(0,(2\pi)^{-1})$. Hence, $y(0) = \Phi(\sqrt{2\pi})-\Phi(-\sqrt{2\pi}) = 2\Phi(\sqrt{2\pi})-1 \approx 0.9826$

2. (a) $R(t) = P\{\mathcal{X}_1 > t\} = \exp(-\lambda t) = (0.999)^t$ while $R_{\text{TMR}}(t) = P\{\mathcal{Y} > t\} = 3((0.999)^t)^2 - 2((0.999)^t)^3 = 3(0.999)^{2t} - 2(0.999)^{3t}$. The two functions both have value 1 at $t = 0$ and are pretty flat for small values of t, though it is always true that $R_{\text{TMR}}(t) > R(t)$ for small $t > 0$. The two curves separate out as t increases, but ultimately for $t > 650$, the curves cross and $R_{\text{TMR}}(t) < R(t)$ for large values of t. Note that the curves *must* cross over since $\mathsf{E}[\mathcal{X}_1] = \int R(t)\,dt > \mathsf{E}[\mathcal{Y}] = \int R_{\text{TMR}}(t)\,dt$ and so it cannot be that $R_{\text{TMR}}(t) > R(t)$ for all $t > 0$.

   (b) $h(t) = \lambda = -\ln 0.999 = 0.00100050\ldots$ is a constant while $h_{\text{TMR}}(t) = \dfrac{6\lambda(0.999)^{2t} - 6\lambda(0.999)^{3t}}{3(0.999)^{2t} - 2(0.999)^{3t}} = 6\lambda\dfrac{1-0.999^t}{3-2(0.999)^t}$ has value 0 at $t = 0$ but asymptotically approaches $2\lambda = 2h(t)$ as $t \to \infty$. In fact, $h_{\text{TMR}}(t)$ exceeds $h(t)$ for $t > 510.5\ldots$.

3. (a) The pdfs are as shown below.



   (b) $\Lambda(u) = \dfrac{f_1(u)}{f_0(u)} = \dfrac{10\cdot\exp(-10u)}{5\cdot\exp(-5u)} = 2\cdot\exp(-5u)$ which has value 2 at $u = 0$ and decays away to 0 as $u \to \infty$. Note that $\Lambda(u) > 1$ for $u < 0.2\ln 2$. Thus, the likelihood ratio test is equivalent to deciding in favor of H_1 if the observed value of $\mathcal{X}$ is *smaller* than the threshold $0.2\ln 2$. Equivalently, $\Gamma_1 = (0, 0.2\ln 2), \Gamma_0 = (0.2\ln 2, \infty)$.

   (c) $$P_{\text{FA}} = \int_{\Gamma_1} f_0(u)\,du = \int_0^{0.5\ln 2} 5\cdot\exp(-5u)\,du = -\exp(-5u)\Big|_0^{0.2\ln 2} = -\frac{1}{2}-(-1) = \frac{1}{2}.$$
   $$P_{\text{MD}} = \int_{\Gamma_0} f_1(u)\,du = \int_{0.5\ln 2}^{\infty} 10\cdot\exp(-10u)\,du = -\exp(-10u)\Big|_{0.2\ln 2}^{\infty} = 0-(-\exp(-2\ln 2)) = \frac{1}{4}.$$

   (d) $\Lambda(u) = 2\cdot\exp(-5u) > \dfrac{\pi_0}{\pi_1}$ for $u < 0.2\ln\left(\dfrac{2\pi_1}{\pi_0}\right) = 0.2\ln 2 + 0.2\ln\left(\dfrac{\pi_1}{\pi_0}\right) = \xi$. Thus, the minimum-error-probability decision rule is equivalent to deciding in favor of H_1 if the observed value of $\mathcal{X}$ is smaller than ξ. Note that $\xi < 0$ if $\pi_0 > 2\pi_1$, that is, if $\pi_0 > 2/3$..

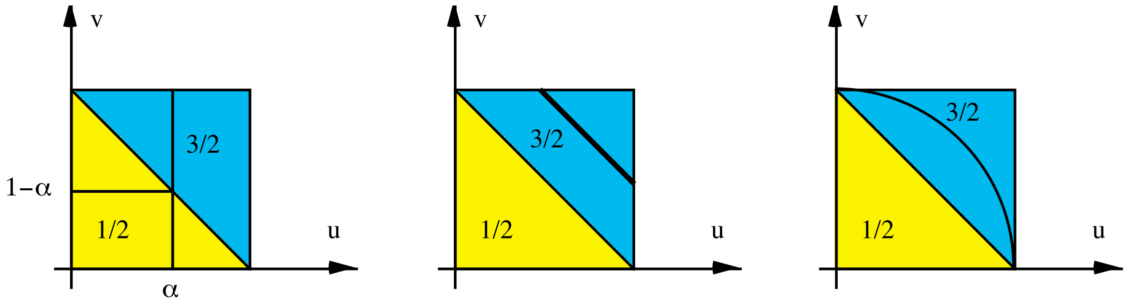
   (e) If $\pi_0 = 1/3$, then $\xi = 0.2\ln 4$. Hence,
   $$P_{\text{FA}} = \int_0^{0.2\ln 4i} 5\cdot\exp(-5u)\,du = -\exp(-5u)\Big|_0^{0.2\ln 4i} = -\frac{1}{4}-(-1) = \frac{3}{4}.$$
   $$P_{\text{MD}} = \int_{0.2\ln 4i}^{\infty} 10\cdot\exp(-10u)\,du = -\exp(-10u)\Big|_{\xi}^{\infty} = 0-(-\exp(-2\ln 4i)) = \frac{1}{16}.$$
   The average error probability thus is $\bar{P}_e = \frac{1}{3}P_{\text{FA}} + \frac{2}{3}_1 P_{\text{MD}} = \frac{7}{24}$. Note that since $\pi_0 < \pi_1$, the Bayesian decision rule allows P_{FA} to increase in return for a decrease in P_{MD} because the latter is weighted more heavily.

   (f) If the decision rule always decides H_1 is the true hypothesis it makes errors if and only if H_0 is the true hypothesis. Hence, $\bar{P}_e = \pi_0$.

(g) When $\pi_0 > 2/3$, the threshold ξ is less than 0. Since $\mathcal{X}$ takes on nonnegative values, it is always larger than the threshold, and hence the decision is always H_0. The average error probability is π_1, and since this is the minimum-error-probability rule, we cannot do any better than this. Note that $\pi_1 < 1/3$.

When $\pi_0 > 2/3$, it follows that $\pi_0 > 2\pi_1$. The average probability of error for the maximum-likelihood rule is $\pi_0 \cdot (1/2) + \pi_1 \cdot (1/4) > 2\pi_1 \cdot (1/2) + \pi_1 \cdot (1/3) = 1.25\pi_1$.

4. The joint pdf is as shown in the figure below.



(a) From the left-hand figure above, $f_{\mathcal{X}}(\alpha) = 0$ for $\alpha < 0$ or $\alpha > 1$, while for any $\alpha, 0 \leq \alpha \leq 1$,

$$f_{\mathcal{X}}(\alpha) = \int_{-\infty}^{\infty} f_{\mathcal{X},\mathcal{Y}}(\alpha, v)\, dv = \int_0^{1-\alpha} \frac{1}{2}\, dv + \int_{1-\alpha}^{1} \frac{3}{2}\, dv = \frac{1}{2}(1-\alpha) + \frac{3}{2}\alpha = \frac{1}{2} + \alpha.$$

(b) When the pdf has constant value over a region, we can find the probability that the random point lies in that region by finding the area of the region and multiplying by the pdf value.

Thus, $P\{\mathcal{X} + \mathcal{Y} \leq 3/2\} = 1 - P\{\mathcal{X} + \mathcal{Y} \geq 3/2\} = 1 - \dfrac{3}{2} \times \left[\dfrac{1}{2} \times \dfrac{1}{2} \times \dfrac{1}{2}\right] = \dfrac{13}{16}$

and $P\{\mathcal{X}^2 + \mathcal{Y}^2 \geq 1\} = \dfrac{3}{2} \times \left[1 - \dfrac{\pi}{4}\right] = \dfrac{3}{2} - \dfrac{3\pi}{8}$.

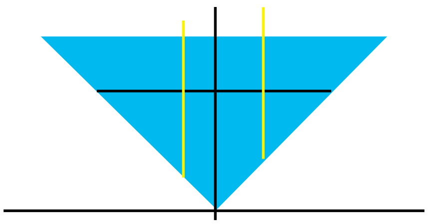
5. The pdf is nonzero over the shaded region in the left-hand figure shown below.

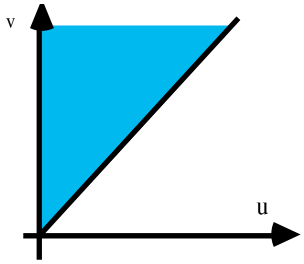
(a) , (**b**) From the figure above, we get that for $v > 0$,

$$f_{\mathcal{Y}}(v) = \int_{u=-v}^{u=+v} c(v^2 - u^2)\exp(-v)\, du = c\left[v^2 u - \frac{u^3}{3}\right]\exp(-v)\Big|_{u=-v}^{u=+v} = c\left(\frac{4}{3}\right) v^3 \exp(-v)$$

which is of the form of a *gamma* pdf with parameters $(4, 1)$. Thus, $4c/3 = 1/\Gamma(4) = 1/3 \Rightarrow c = 1/8$. On the other hand, for $u > 0$, $f_{\mathcal{X}}(u) = \displaystyle\int_{v=u}^{\infty} \frac{1}{8}(v^2 - u^2)\exp(-v)\, dv = \frac{1}{4}(1+u)\exp(-u)$, while if $u < 0$, the limits are $v = -u$ and ∞. Consequently, $f_{\mathcal{X}}(u) = \frac{1}{4}(1 + |u|)\exp(-u), -\infty < u < \infty$.

(c) The pdf of $\mathcal{X}$ is an even function of u and $\displaystyle\int_0^{\infty} f_{\mathcal{X}}(u)\, du$ is finite. Hence, $\mathsf{E}[\mathcal{X}] = 0$.





6. (a) The joint pdf is nonzero on the shaded region shown in the right-hand figure above.

(b) For $u > 0$, $f_{\mathcal{X}}(u) = \int_{v}^{\infty} 2\exp(-u-v)\,dv = 2\exp(-2u)$ and $f_{\mathcal{X}}(u) = 0$ for $u < 0$.

For $v > 0$, $f_{\mathcal{Y}}(v) = \int_{0}^{v} 2\exp(-u-v)\,du = 2\exp(-2v)$ and $f_{\mathcal{Y}}(v) = 0$ for $v < 0$.

Note that both marginal pdfs are exponential with parameter 2.

(c) No, the eyeball test says that the random variables are dependent.

(d) $P\{\mathcal{Y} > 3\mathcal{X}\} = \int_{u=0}^{\infty}\int_{v=3u}^{\infty} 2(\exp(-u-v)\,dv\,du = \int_{u=0}^{\infty} 2\exp(-4u)\,du = \frac{1}{2}$.

(e) For $\alpha > 0$,

$$P\{\mathcal{X}+\mathcal{Y} \leq \alpha\} = \int_{u=0}^{\alpha/2}\int_{v=0}^{\alpha} 2e^{-u-v}\,dv\,du = \int_{u=0}^{\alpha/2} 2e^{-u}[e^{-u)} - e^{-\alpha+u)}]\,du = (1+\alpha)\exp(-\alpha).$$

(f) $f_{\mathcal{Z}}(\alpha) = \frac{d}{d\alpha}F_{\mathcal{Z}}(\alpha) = \frac{d}{d\alpha}P\{\mathcal{X}+\mathcal{Y} \leq \alpha\} = \frac{d}{d\alpha}(1+\alpha)\exp(-\alpha) = \alpha\exp(-\alpha)$ for $\alpha > 0$ and $f_{\mathcal{Z}}(\alpha) = 0$ for $\alpha < 0$. This is a *gamma* pdf wiith parameters $(2,1)$.