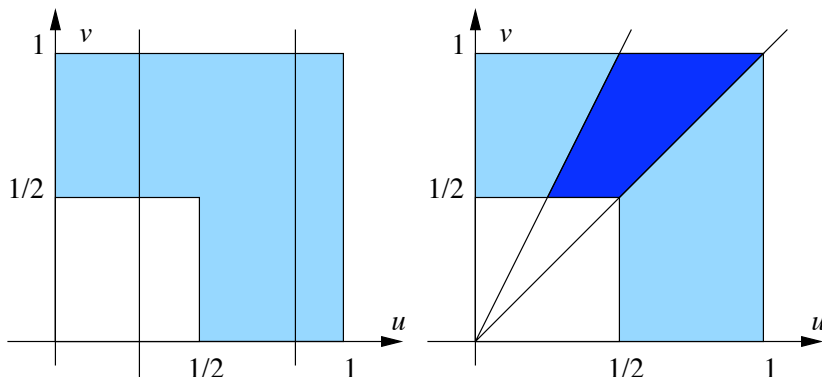


ECE 413: Solutions to Problem Set 13

1. The random point $(\mathcal{X}, \mathcal{Y})$ is uniformly distributed on the shaded region shown in the figures below. Clearly, $f_{\mathcal{X}, \mathcal{Y}}(u, v) = \frac{4}{3}$ on this region.



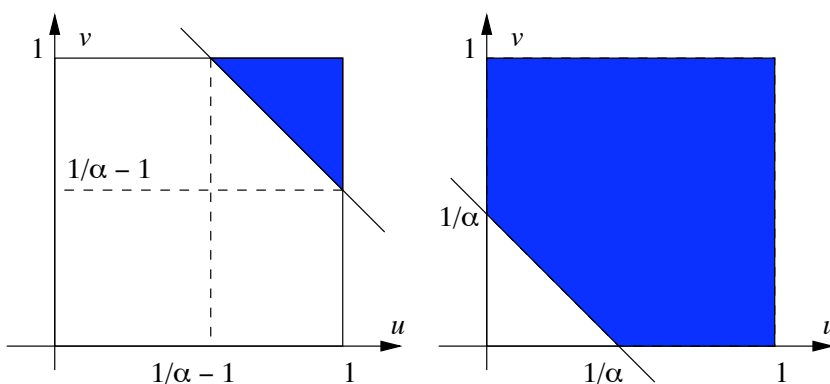
- (a) $f_{\mathcal{X}}(u)$ is the area of the cross-section of the pdf surface along the line u . There are two cases to be considered, as shown in the left-hand figure above. It is obvious almost by inspection that

$$f_{\mathcal{X}}(u) = \begin{cases} \frac{2}{3}, & 0 \leq u \leq \frac{1}{2}, \\ \frac{4}{3}, & \frac{1}{2} < u \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

- (b) By symmetry, $f_{\mathcal{X}}$ and $f_{\mathcal{Y}}$ are the same function: $f_{\mathcal{Y}}(v) = \begin{cases} \frac{2}{3}, & 0 \leq v \leq \frac{1}{2}, \\ \frac{4}{3}, & \frac{1}{2} < v \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$

- (c) $P\{\mathcal{X} < \mathcal{Y} < 2\mathcal{X}\}$ is the volume in the deep-shaded trapezoidal region in the right-hand figure above. The area of the region is $\frac{1}{2} \left[\frac{1}{2} + \frac{1}{4} \right] \frac{1}{2} = \frac{3}{16}$ and hence $P\{\mathcal{X} < \mathcal{Y} < 2\mathcal{X}\} = \frac{3}{16} \times \frac{4}{3} = \frac{1}{4}$.

2. $\mathcal{I} = 1/(\mathcal{R}_1 + \mathcal{R}_2)$ takes on values in $(1/2, \infty)$.



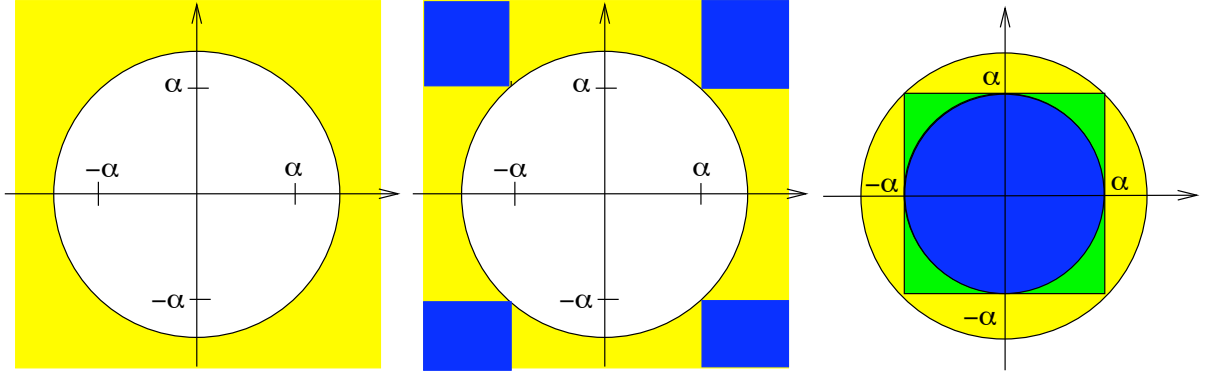
Now, $F_{\mathcal{I}}(\alpha) = P\{\mathcal{I} \leq \alpha\} = P\{\mathcal{R}_1 + \mathcal{R}_2 \geq \alpha^{-1}\}$. For $1/2 \leq \alpha \leq 1$, we see from the left-hand figure above that

$$F_{\mathcal{I}}(\alpha) = \frac{1}{2}(1 - (\alpha^{-1} - 1))^2 = \frac{1}{2}(2 - \alpha^{-1})^2.$$

For $\alpha > 1$, we see from the right-hand figure above that $F_{\mathcal{I}}(\alpha) = 1 - \frac{1}{2}(\alpha^{-1})^2$.

Consequently, $f_{\mathcal{I}}(\alpha) = \begin{cases} 2\alpha^{-2} - \alpha^{-3}, & \frac{1}{2} \leq \alpha \leq 1, \\ \alpha^{-3}, & \alpha > 1, \\ 0, & \text{elsewhere.} \end{cases}$

3. (a) $f_{\mathcal{X},\mathcal{Y}}(u,v) = f_{\mathcal{X}}(u)f_{\mathcal{Y}}(v) = \frac{1}{2\pi} \exp\left[-\frac{u^2+v^2}{2}\right]$.
- (b) The region over which the joint pdf must be integrated in order to find $P\{\mathcal{X}^2 + \mathcal{Y}^2 > 2\alpha^2\}$ is shown in the left-hand figure below.



The volume outside the circle of radius $\sqrt{2}\alpha$ can be found by changing to polar coordinates. We have $P\{\mathcal{X}^2 + \mathcal{Y}^2 > 2\alpha^2\} = \int_{r=\sqrt{2}\alpha}^{\infty} \int_{\theta=0}^{2\pi} \frac{1}{2\pi} \exp\left(-\frac{r^2}{2}\right) r d\theta dr = \int_{r=\sqrt{2}\alpha}^{\infty} r \exp\left(-\frac{r^2}{2}\right) dr = -\exp\left(-\frac{r^2}{2}\right) \Big|_{\sqrt{2}\alpha}^{\infty} = \exp(-\alpha^2)$.

- (c) $\mathcal{Z} = \mathcal{X}^2 + \mathcal{Y}^2$. From part (b), we get that $P\{\mathcal{Z} > \beta\} = 1 - F_{\mathcal{Z}}(\beta) = \exp(-\beta/2)$ for $\beta \geq 0$. Hence, $f_{\mathcal{Z}}(\beta) = \frac{1}{2} \exp(-\beta/2)$ for $\beta \geq 0$. This is an exponential density with parameter $\frac{1}{2}$.
- (d) $P\{|\mathcal{X}| > \alpha\} = 2Q(\alpha)$. Hence, $P\{|\mathcal{X}| > \alpha, |\mathcal{Y}| > \alpha\} = P\{|\mathcal{X}| > \alpha\}P\{|\mathcal{Y}| > \alpha\} = 4Q^2(\alpha)$.
- (e) From the middle figure above, it is obvious that

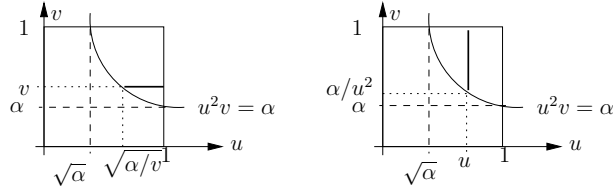
$$P\{|\mathcal{X}| > \alpha, |\mathcal{Y}| > \alpha\} = 4Q^2(\alpha) < P\{\mathcal{X}^2 + \mathcal{Y}^2 > 2\alpha^2\} = \exp(-\alpha^2) \text{ for } \alpha > 0.$$

- (f) Taking square roots on both sides, we get $Q(x) < \frac{1}{2} \exp(-x^2/2)$ for $x > 0$.
- (g) $P\{|\mathcal{X}| < \alpha, |\mathcal{Y}| < \alpha\} = P\{|\mathcal{X}| < \alpha\}P\{|\mathcal{Y}| < \alpha\} = [1 - 2Q(\alpha)]^2$ is the probability that the random point lies inside the square of side 2α . As shown in the right-hand figure, this square is inscribed by the circle of radius α and circumscribed by the circle of radius $\sqrt{2}\alpha$. Hence,

$$P\{\mathcal{X}^2 + \mathcal{Y}^2 \leq \alpha^2/2\} < P\{|\mathcal{X}| < \alpha, |\mathcal{Y}| < \alpha\} < P\{\mathcal{X}^2 + \mathcal{Y}^2 < \alpha^2\}.$$

These inequalities are equivalent to the result: $\exp(-\alpha^2) < 4Q(\alpha) - 4Q^2(\alpha) < \exp(-\alpha^2/2)$. Now, $4Q(\alpha) - 4Q^2(\alpha) < 4Q(\alpha)$, and therefore we get $Q(x) > \frac{1}{4} \exp(-x^2)$ for $x > 0$. Note that $Q(0) = \frac{1}{2}$ while the bound equals $\frac{1}{4}$ at $x = 0$. The upper bound can also be obtained from the above result. Note that since $Q(\alpha) \leq \frac{1}{2}$ for $\alpha > 0$, it follows that $4Q(\alpha) - 4Q^2(\alpha) = 4Q(\alpha)[1 - Q(\alpha)] > 2Q(\alpha)$ for $\alpha > 0$. Hence, $\exp(-\alpha^2/2) > 4Q(\alpha) - 4Q^2(\alpha) > 2Q(\alpha)$ etc.

4. $\mathcal{Z} = \mathcal{X}^2\mathcal{Y}$ takes on values between 0 and 1.



For $0 < \alpha < 1$,

$$P\{\mathcal{Z} > \alpha\} = \int_{v=\alpha}^1 \int_{u=\sqrt{\alpha/v}}^1 2u \, du \, dv = \int_{v=\alpha}^1 u^2 \Big|_{\sqrt{\alpha/v}}^1 \, dv = \int_{v=\alpha}^1 \left(1 - \frac{\alpha}{v}\right) \, dv = v - \alpha \ln(v) \Big|_{\alpha}^1$$

$$= 1 - \alpha + \alpha \ln(\alpha). \text{ Hence, } f_{\mathcal{Z}}(\alpha) = -\frac{d}{d\alpha} P\{\mathcal{Z} > \alpha\} = \begin{cases} -\ln(\alpha), & 0 < \alpha < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{Alternatively, } P\{\mathcal{Z} > \alpha\} = \int_{u=\sqrt{\alpha}}^1 \int_{v=\alpha/u^2}^1 2u \, dv \, du = \int_{u=\sqrt{\alpha}}^1 2u \left(1 - \frac{\alpha}{u^2}\right) \, du = u^2 - 2\alpha \ln(u) \Big|_{\sqrt{\alpha}}^1$$

$$= 1 - \alpha + \alpha \ln(\alpha) \text{ as before. . .}$$

5. The marginal pdf of \mathcal{X} has value $f_{\mathcal{X}}(u) = \begin{cases} \frac{2}{3}, & 0 \leq u \leq \frac{1}{2}, \\ \frac{4}{3}, & \frac{1}{2} < u \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$

$$(a) \quad E[\mathcal{X}] = \int_0^{1/2} u \cdot \frac{2}{3} \, du + \int_{1/2}^1 u \cdot \frac{4}{3} \, du = \frac{u^2}{3} \Big|_0^{1/2} + \frac{2u^2}{3} \Big|_{1/2}^1 = \frac{1}{12} + \frac{2}{3} - \frac{1}{6} = \frac{7}{12}.$$

$$E[\mathcal{X}^2] = \int_0^{1/2} u^2 \cdot \frac{2}{3} \, du + \int_{1/2}^1 u^2 \cdot \frac{4}{3} \, du = \frac{2u^3}{9} \Big|_0^{1/2} + \frac{4u^3}{9} \Big|_{1/2}^1 = \frac{1}{36} + \frac{4}{9} - \frac{1}{18} = \frac{15}{36} = \frac{5}{12}.$$

$$\text{var}(\mathcal{X}) = E[\mathcal{X}^2] - (E[\mathcal{X}])^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{60}{144} - \frac{49}{144} = \frac{11}{144}.$$

(b) Since \mathcal{Y} has the same pdf as \mathcal{X} , it has the same mean and variance as \mathcal{X} .

$$(c) \quad E[\mathcal{X}\mathcal{Y}] = \int_0^1 \int_0^1 uv \frac{4}{3} \, du \, dv - \int_0^{1/2} \int_0^{1/2} uv \frac{4}{3} \, du \, dv = \frac{4}{3} \int_0^1 u \, du \int_0^1 v \, dv - \frac{4}{3} \int_0^{1/2} u \, du \int_0^{1/2} v \, dv$$

$$= \frac{4}{3} \left[\frac{1}{2} \times \frac{1}{2} - \frac{1}{8} \times \frac{1}{8} \right] = \frac{5}{16}. \text{ Hence, } \text{cov}(\mathcal{X}, \mathcal{Y}) = E[\mathcal{X}\mathcal{Y}] - E[\mathcal{X}]E[\mathcal{Y}] = \frac{5}{16} - \frac{49}{144} = -\frac{4}{144} = -\frac{1}{36}.$$

6. $E[\mathcal{X}] = 1$, $E[\mathcal{Y}] = 4$, $\text{var}(\mathcal{X}) = 4$, $\text{var}(\mathcal{Y}) = 9$, and $\rho_{\mathcal{X}, \mathcal{Y}} = 0.1$.

$$(a) \quad E[\mathcal{Z}] = E[2(\mathcal{X} + \mathcal{Y})(\mathcal{X} - \mathcal{Y})] = 2E[\mathcal{X}^2 - \mathcal{Y}^2] = 2E[\mathcal{X}^2] - 2E[\mathcal{Y}^2] = 2[4 + 1^2] - 2[9 + 4^2] = -40.$$

$$(b) \quad \text{cov}(\mathcal{T}, \mathcal{U}) = \text{cov}(2\mathcal{X} + \mathcal{Y}, 2\mathcal{X} - \mathcal{Y}) = 4 \cdot \text{cov}(\mathcal{X}, \mathcal{X}) + 2 \cdot \text{cov}(\mathcal{Y}, \mathcal{X}) - 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y}) - \text{cov}(\mathcal{Y}, \mathcal{Y})$$

$$= 4 \cdot \text{var}(\mathcal{X}) + 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y}) - 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y}) - \text{var}(\mathcal{Y}) = 4 \cdot \text{var}(\mathcal{X}) - \text{var}(\mathcal{Y}) = 4 \cdot 4 - 9 = 7.$$

$$(c) \quad E[\mathcal{W}] = E[3\mathcal{X} + \mathcal{Y} + 2] = 3E[\mathcal{X}] + E[\mathcal{Y}] + 2 = 9.$$

$$\text{var}(\mathcal{W}) = \text{var}(3\mathcal{X} + \mathcal{Y} + 2) = 3^2 \cdot \text{var}(\mathcal{X}) + \text{var}(\mathcal{Y}) + 2 \cdot 3 \cdot 1 \cdot \text{cov}(\mathcal{X}, \mathcal{Y}) = 9 \cdot 4 + 9 + 6 \cdot 2 \cdot 3 \cdot 0.1 = 48.6.$$

$$(d) \quad P\{\mathcal{W} > 0\} = 1 - \Phi\left(\frac{0-9}{\sqrt{48.6}}\right) = 1 - \Phi\left(-\frac{9}{\sqrt{48.6}}\right) = \Phi\left(\frac{9}{\sqrt{48.6}}\right).$$

7. (a) $\text{var}(\mathcal{X} + \mathcal{Y}) = \text{var}(\mathcal{X}) + \text{var}(\mathcal{Y}) + 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y}) = 36$.

$$\text{var}(\mathcal{X} - \mathcal{Y}) = \text{var}(\mathcal{X}) + \text{var}(\mathcal{Y}) - 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y}) = 64. \text{ Hence, } \text{cov}(\mathcal{X}, \mathcal{Y}) = -7.$$

$$\text{From the above, } 2 \cdot \text{var}(\mathcal{X}) + 2 \cdot \text{var}(\mathcal{Y}) = 8 \cdot \text{var}(\mathcal{Y}) = 100, \text{ giving } \text{var}(\mathcal{Y}) = 12.5, \text{var}(\mathcal{X}) = 37.5 \text{ and}$$

$$\rho_{\mathcal{X}, \mathcal{Y}} = \text{cov}(\mathcal{X}, \mathcal{Y}) / \sqrt{\text{var}(\mathcal{X})\text{var}(\mathcal{Y})} = -7/12.5\sqrt{3}.$$

(b) $\text{var}(\mathcal{X} + \mathcal{Y}) = \text{var}(\mathcal{X}) + \text{var}(\mathcal{Y}) + 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y})$ equals $\text{var}(\mathcal{X} - \mathcal{Y}) = \text{var}(\mathcal{X}) + \text{var}(\mathcal{Y}) - 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y})$ if and only if $\text{cov}(\mathcal{X}, \mathcal{Y}) = 0$, that is, if and only if \mathcal{X} and \mathcal{Y} are uncorrelated.

(c) No, whether $\text{var}(\mathcal{X})$ equals $\text{var}(\mathcal{Y})$ or not has no bearing on the question of whether $\text{cov}(\mathcal{X}, \mathcal{Y})$ is zero or not.