

ECE 413: Problem Set 1

Due: Wednesday August 30 at the beginning of class.

Reading: Ross, Chapters 1 and 2

This Problem Set contains five problems

Note: Most of the topics covered on this problem set will be needed in ECE 413 after the drop deadline. Thus this problem set (which is based entirely on material covered in the prerequisites to ECE 413) is intended as a review and a diagnostic aid to help you identify topics that might need reviewing before embarking on ECE 413.

1. (a) Prove that $1 + x + x^2 + \dots + x^{n-1} = \frac{1 - x^n}{1 - x}$ for all $x \neq 1$ and all integers $n \geq 1$.
- (b) Assuming that $|x| < 1$, find the sum of the series $1 + x + x^2 + \dots$.
Hint: it is the limit of the finite sum $1 + x + x^2 + \dots + x^{n-1}$ as $n \rightarrow \infty$.
- (c) For $0 < k \leq n$, compute the k -th derivative of $f(x) = (1 + x)^n$ using the chain rule, that is, *without* multiplying out the terms to get a polynomial in x . Use these derivatives to find the first $n + 1$ terms of the MacLaurin series (Taylor series in the vicinity of 0) for $(1 + x)^n$.
- (d) Repeat part (c) for $k > n$ and thus find the complete Maclaurin series for $f(x)$.
- (e) According to the textbook (Equation 4.2 in Chapter 1 with $y = 1$),

$$(1 + x)^n = \sum_{k=0}^n \binom{n}{k} x^k \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Does your answer to part (d) match this result? If so, congratulations! You have just proved the binomial theorem for positive integer exponents.

- (f) Now consider the function $g(x) = (1 - x)^{-n}$ where n is a positive integer. Does the MacLaurin series for $g(x)$ contain terms of degree $> n$? If so, what is the term for degree $n + 1$? If not, what is the highest degree term?
 - (g) Use the result of part (f) to write down the MacLaurin series for $(1 - x)^{-1}$ and $(1 - x)^{-2}$. These results together with the one of parts (d) and (e) will be needed so often in ECE 413 that it is recommended that you memorize them.
 - (h) Find the MacLaurin series for $(1+x)^a$ where a is a real number and not necessarily an integer.
2. (a) Find the limit of the function $\frac{1}{\sin^2 x} - \frac{1}{x^2}$ as $x \rightarrow 0$.
 - (b) Find the maximum value of $x^{25}(1.00001)^{-x}$ in the interval $(0, \infty)$.

3. (a) Evaluate the integrals $\int_{-1}^2 |x| dx$ and $\int_{-2}^1 x(1-x)^{19} dx$.
- (b) Does there exist a *nonnegative* function $f(x)$ such that $\int_{-2}^1 f(x) dx < 0$. [Hint: Does either function of part (a) satisfy these conditions?]
- (c) Let $f(x)$ denote a differentiable function with derivative $g(x)$. Which of the following statements are true? In parts (v) and (vi), C denotes an arbitrary constant.
- (i) $\frac{d}{dx} f(-x) = -g(x)$ (ii) $\frac{d}{dx} f(x^2) = g(x^2)$
- (iii) $\frac{d}{dx} \exp(f(x)) = \exp(f(x))g(x)$ (iv) $\frac{d}{dx} \exp(f(x^2)) = \exp(f(x))g(x^2)$
- (v) $\int g(-x) dx = f(-x) + C$ (vi) $\int g(x^2) dx = f(x^2)/(2x) + C$
- (d) Let $I = \int_{-1}^1 \frac{2}{1+x^2} dx$ and $J = \int_{-1}^1 \frac{-2}{1+y^2} dy$, and note that $I = -J$.
- i. Show that $I = \pi$. [Hint: the integrand is the derivative of $\arctan(x)$.]
 ii. Use the substitution $y = 1/x$ to transform I into J .
 iii. $I = J$ and $I = -J$ if and only if $I = J = 0$. Does this mean that $\pi = 0$?
4. (a) Find the integral of $f(x, y) = \max(x, y)$ over $\{(x, y) : 0 < x < 2, 0 < y < 1\}$.
 (b) Find the integral of $(x^2 + y^2)^{-2}$ over $\{(x, y) : x^2 + y^2 > 4\}$.
5. (a) What is the derivative of $\exp(-x^2/2)$?
 (b) Evaluate $\int_0^\infty x \exp(-x^2/2) dx$.