University of Illinois Fall 2006

ECE 413: Problem Set 9

Due: Wednesday November 1 at the beginning of class.
Reading: Ross, Chapters 4 and 5
Reminder: No class on Friday October 27
Cancelled class will be made up on Monday October 30, 7-8 pm, 169 EL

This Problem Set contains six problems

1. Two discrete random variables $\mathcal{X}$ and $\mathcal{Y}$ respectively taking on values $\{u_1, u_2, \ldots\}$ and $\{v_1, v_2, \ldots\}$ are called (unconditionally) *independent* random variables if for *all* u_i and v_j,

$$P[\{\mathcal{X} = u_i\} \cap \{\mathcal{Y} = v_j\}] = P[\{\mathcal{X} = u_i\}]P[\{\mathcal{Y} = v_j\}].$$

Equivalently, the *conditional pmf* of $\mathcal{Y}$ given that $\mathcal{X} = u_i$ is the same for all u_i. The random variables are *conditionally independent* given an event A if *for all* u_i and v_j,

$$P[\{\mathcal{X} = u_i\} \cap \{\mathcal{Y} = v_j\}|A] = P[\{\mathcal{X} = u_i\}|A]P[\{\mathcal{Y} = v_j\}|A].$$

Note that A might be specified in terms of another random variable.

With this as prologue, let us return to Problem 1 of Problem Set 8. Let $\mathcal{W}$ denote the number of α-particles that are *not* detected by the Geiger counter. Note that given that $\{\mathcal{X} = n\}$, $\mathcal{W} = n - \mathcal{Y}$ has conditional pmf $\mathsf{Binomial}(n, 1-p)$ and its unconditional pmf is $\mathsf{Poisson}(\lambda(1-p))$.

(a) Explain why $P[\{\mathcal{Y} = k\} \cap \{\mathcal{W} = l\}|\{\mathcal{X} = n\}] = 0$ except when $n = k + l$ when this probability is $\binom{k+l}{k}p^k(1-p)^l$.

(b) Are $\mathcal{Y}$ and $\mathcal{W}$ conditionally independent given the event $\{\mathcal{X} = n\}$?

(c) Use the theorem of total probability in conjunction with your answer of part (a) to write down the value of $P[\{\mathcal{Y} = k\} \cap \{\mathcal{W} = l\}]$.

(d) Are $\mathcal{Y}$ and $\mathcal{W}$ *unconditionally* independent?

2. Which of the following functions $F(u)$ are valid CDFs? For those that are valid CDFs, compute the probability that the absolute value of the random variable exceeds 0.5.

(a) $F(u) = \begin{cases} 0 & u < 0, \\ u^2, & 0 \le u < 1, \\ 1, & u \ge 1. \end{cases}$ (b) $F(u) = \begin{cases} 0 & u < 1, \\ 2u - u^2, & 1 \le u \le 2, \\ 1, & u > 2. \end{cases}$

(c) $F(u) = \begin{cases} \frac{1}{2}\exp(2u) & u \le 0, \\ 1 - \frac{1}{4}\exp(-3u), & u > 0, \end{cases}$ (d) $F(u) = \begin{cases} \frac{1}{2}\exp(2u) & u < 0, \\ 1 - \frac{1}{4}\exp(-3u), & u \ge 0, \end{cases}$

3. The number of hours that a student spends on ECE 440 homework is a random variable $\mathcal{X}$ with CDF

$$F_{\mathcal{X}}(u) = \begin{cases} 0, & u < 0, \\ (1+u)/8, & 0 \le u < 1, \\ 1/2, & 1 \le u < 2, \\ (4+u)/8, & 2 \le u < 4, \\ 1, & u \ge 4. \end{cases}$$

Note that this is a *mixed* random variable: it takes on some values with nonzero probability (like a discrete random variable) but also takes on all values in intervals of the real line (like a continuous random variable).

(a) Find $P\{\mathcal{X} = 2\}$, $P\{\mathcal{X} < 2\}$, $P\{\mathcal{X} > 2\}$, $P\{1 \leq \mathcal{X} \leq 3\}$, and $P\{\mathcal{X} > 2 \mid \mathcal{X} > 0\}$.

(b) Find $\mathsf{E}[\mathcal{X}]$.

4. The expectation of a nonnegative random variable $\mathcal{X}$ is

$$\mathsf{E}[\mathcal{X}] = \int_0^\infty [1 - F_\mathcal{X}(u)]\, du = \int_0^\infty P\{\mathcal{X} > u\} du.$$

(a) Use this result to prove that if $\mathcal{X}$ is a discrete random variable that takes on nonnegative *integer* values, then $\mathsf{E}[\mathcal{X}] = \sum_{k=0}^{\infty} P\{X > k\} = \sum_{i=1}^{\infty} P\{X \geq i\}$. (cf. Theoretical Exercise 6, p. 197 of Ross).

(b) For $k = 0, 1, 2, \ldots$, find $P\{\mathcal{X} > k\}$ for a *geometric* random variable with parameter p. Use these results together with the result of part (a) to provide a different proof of the fact that $\mathsf{E}[\mathcal{X}] = p^{-1}$.

(c) Theoretical Exercise 7 on pp. 197-198 of Ross.

5. Nine functions $f(u)$ are shown below. Note that in each case, $f(u) = 0$ for all u not in the interval specified. In each case,

- determine whether $f(u)$ is a valid probability density function (pdf).
- If $f(u)$ is not a valid pdf, determine if there exists a constant C such that $C \cdot f(u)$ is a valid pdf.

(a) $f(u) = 2u, \quad 0 < u < 1.$

(b) $f(u) = |u|, \quad |u| < \frac{1}{2}$

(c) $f(u) = 1 - |u|, \quad |u| < 1,$

(d) $f(u) = \ln u, \quad 0 < u < 1.$ Hint: $\ln u$ can be integrated by parts.

(e) $f(u) = \ln u, \quad 0 < u < 2,$

(f) $f(u) = \frac{2}{3}(u - 1), \quad 0 < u < 3,$

(g) $f(u) = \exp(-2u), \quad u > 0.$

(h) $f(u) = 4\exp(-2u) - \exp(-u), \quad u > 0,$

(i) $f(u) = \exp(-|u|), \quad |u| < 1,$

6. Problem 4 on page 247 of Ross.