University of Illinois Fall 2006

ECE 413: Problem Set 11

Due: Wednesday November 15 at the beginning of class.
Reading: Ross, Chapter 5
Reminder: The Second Hour Exam is scheduled for
Monday November 13, 7:00 p.m. – 8:00 p.m.
Room 119 Materials Science Building.
You are allowed to bring one $8\frac{1}{2}$" × 11" sheet of notes with you, but the exam is closed-book, closed-notes otherwise. Electronic devices are not permitted.
Your notesheet should have on it the pmfs/pdfs and means and variances of the following random variables: *discrete:* binomial, Poisson, geometric, and negative binomial; *continuous:* uniform, exponential, Gaussian a.k.a. normal
Coverage on the Exam is Conditional Probability and Independent Events, Decision-Making, and Continuous Random Variables (Chapter 5 of Ross *except* Sections 5.5.1 and 5.7)

This Problem Set contains six problems

1. The weekly demand (measured in thousands of gallons) for gasoline at a rural gas station is a random variable X with probability density function

$$f_{\mathcal{X}}(u) = \begin{cases} 5(1-u)^4, & 0 \le u \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

   Let C (in thousands of gallons) denote the capacity of the tank (which is re-filled weekly.)

   (a) If $C = 0.5$, what is the probability that the weekly demand for gasoline can be satisfied? Note that if your answer is (say) 0.666..., then, in the long run, the gas station can supply the weekly demand two weeks out of three.

   (b) What is the minimum value of C required to ensure that the probability that the demand exceeds the supply is no larger than 10^{-5} ?

   (c) Suppose that the owner makes a gross profit of \$0.64 for each gallon of gasoline sold. Let $\mathcal{Y}$ denote the amount of gasoline sold per week. How is $\mathcal{Y}$ related to $\mathcal{X}$, the weekly demand for gasoline? (Hint: the owner cannot sell more gasoline each week than the tank can hold!) What is the *average* weekly gross profit?

   (d) Suppose that the owner pays \$$20C$ as weekly rent on a tank of capacity $1000C$ gallons. Note that $0 \le C \le 1$. (Why is a tank larger than 1000 gallons not needed?) What is the average weekly net profit and what value of C maximizes the average weekly net profit?

2. The current I through a semiconductor diode is related to the voltage V across the diode as $I = I_0(\exp(V) - 1)$ where I_0 is the magnitude of the reverse current. Suppose that the voltage across the diode is modeled as a continuous random variable $\mathcal{V}$ with pdf

$$f_{\mathcal{V}}(u) = 0.5\exp(-|u|), -\infty < u < \infty.$$

   Then, the current $\mathcal{I}$ is also a continuous random variable.

   (a) What values can $\mathcal{I}$ take on?

   (b) Find the CDF of $\mathcal{I}$.

   (c) Find the pdf of $\mathcal{I}$.

3. $\mathcal{X}$ is uniformly distributed on $[-1, +1]$.

   (a) Find the pdf of $\mathcal{Y} = \mathcal{X}^2$.

   (b) Find the pdf of $\mathcal{Z} = g(\mathcal{X})$ where $g(u) = \begin{cases} u^2, & u \geq 0 \\ -u^2, & u < 0 \end{cases}$.

4. [Give me an A! Give me a D! Give me a converter! What have we got? An A/D converter! Go Team!] A signal $\mathcal{X}$ is modeled as a unit Gaussian random variable. For some applications, however, only the quantized value $\mathcal{Y}$ (where $\mathcal{Y} = \alpha$ if $\mathcal{X} > 0$ and $\mathcal{Y} = -\alpha$ if $\mathcal{X} \leq 0$) is used. Note that $\mathcal{Y}$ is a *discrete* random variable.

   (a) What is the pmf of $\mathcal{Y}$?

   (b) The *squared error* in representing $\mathcal{X}$ by $\mathcal{Y}$ is $\mathcal{Z} = \begin{cases} (\mathcal{X} - \alpha)^2, & \text{if } \mathcal{X} > 0, \\ (\mathcal{X} + \alpha)^2, & \text{if } \mathcal{X} \leq 0, \end{cases}$ and varies as different trials of the experiment produce different values of $\mathcal{X}$. We would like to choose the value of α so as to minimize the *mean* squared error $\mathsf{E}[\mathcal{Z}]$. Use LOTUS to ez-ily calculate $\mathsf{E}[\mathcal{Z}]$ (the answer will be a function of α), and then find the value of α that minimizes $\mathsf{E}[\mathcal{Z}]$.

   (c) We now get more ambitious and use a 3-bit A/D converter which first quantizes $\mathcal{X}$ to the nearest integer $\mathcal{W}$ in the range 3 to +3. Thus, $\mathcal{W} = 3$ if $\mathcal{X} \geq 2.5$, $\mathcal{W} = 2$ if $1.5 \leq \mathcal{X} < 2.5$, $\mathcal{W} = 1$ if $0.5 \leq \mathcal{X} < 1.5$, $\cdots$, $\mathcal{W} = -3$ if $\mathcal{X} < -2.5$. Note that $\mathcal{W}$ is also a discrete random variable. Find the pmf of $\mathcal{W}$.

   (d) The output of the A/D converter is a 3-bit 2's complement representation of $\mathcal{W}$. Suppose that the output is $(\mathcal{Z}_2, \mathcal{Z}_1, \mathcal{Z}_0)$. What is the pmf of $\mathcal{Z}_2$? the pmf of $\mathcal{Z}_1$? the pmf of $\mathcal{Z}_0$? Note that $(1, 0, 0)$ which represents -4 is not one of the possible outputs from this A/D converter.

5. Let $\mathcal{X}$ denote the time of the first arrival after $t = 0$ in a Poisson process with arrival rate λ.

   (a) What is the value of the CDF of $\mathcal{X}$ at time T? that is, what is $P\{\mathcal{X} \leq T\}$?

   (b) Let A denote the event that there is exactly one arrival in the interval $(0, T]$. What is $P(A)$?

   (c) Is the $P(A)$ that you found for part (b) the same as the value of $P\{\mathcal{X} \leq T\}$ that you gave in part (a) ? Explain why the two are the same (or are different, as appropriate).

   (d) For a specific fixed real number t such that $0 < t < T$, what is the conditional probability that $\{X \leq t\}$ given the event A, that is, given that there was exactly one arrival in $(0, T]$?

6. Consider a Poisson process with arrival rate λ.

   (a) What is the mean number of arrivals in the interval $(0, 4]$? That is, what is $\mathsf{E}[N(0, 4]]$?

   (b) What is $P[\{N(0, 3] = 3\} \cap \{N(2, 6] = 0\}]$?

   (c) If we observe that there were 5 arrivals in $(0, 6]$, what is the maximum-likelihood estimate of the arrival rate λ?

   (d) Now suppose that $\lambda = \ln 2$. What is the probability that at least one arrival occurs in $(0, t]$?