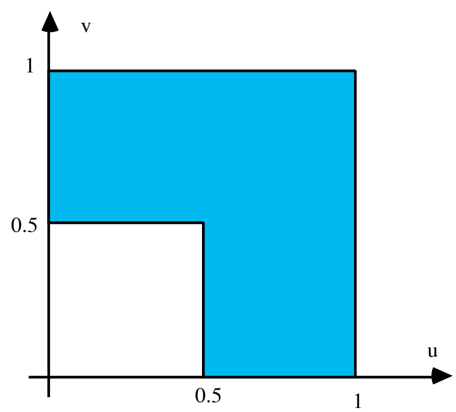
University of Illinois Fall 2006

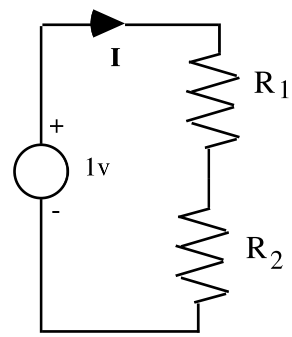
ECE 413: Problem Set 13

Due: Friday May 8 at the beginning of class.
Reading: Ross, Chapter 6 and 7

This Problem Set contains seven problems

1. The random point $(\mathcal{X}, \mathcal{Y})$ is uniformly distributed on the shaded region shown in the left hand figure below.
   (a) Find the marginal pdf $f_{\mathcal{X}}(u)$ of the random variable $\mathcal{X}$.
   (b) *Write down* the marginal pdf $f_{\mathcal{Y}}(v)$ of the random variable $\mathcal{Y}$ from your answer to part (a).
   (c) Find $P\{\mathcal{X} < \mathcal{Y} < 2\mathcal{X}\}$.





2. Two resistors are connected in series to a one-volt voltage source as shown in the righthand diagram above. Suppose that the resistance values $\mathcal{R}_1$ and $\mathcal{R}_2$ (measured in ohms) are independent random variables, each uniformly distributed on the interval $(0, 1)$. Find the pdf $f_{\mathcal{I}}(a)$ of the current $\mathcal{I}$ (measured in amperes) in the circuit.

3. $\mathcal{X}$ and $\mathcal{Y}$ denote *independent* standard Gaussian random variables.
   (a) What is the joint pdf $f_{\mathcal{X},\mathcal{Y}}(u, v)$ of $\mathcal{X}$ and $\mathcal{Y}$?
   (b) Sketch the u-v plane and indicate on it the region over which you need to integrate the joint pdf in order to find $P\{\mathcal{X}^2 + \mathcal{Y}^2 > 2\alpha^2\}$. Compute $P\{\mathcal{X}^2 + \mathcal{Y}^2 > 2\alpha^2\}$.
   (c) Let $\mathcal{Z} = \mathcal{X}^2 + \mathcal{Y}^2$. What is the pdf of $\mathcal{Z}$?
   (d) Express $P\{|\mathcal{X}| > \alpha\}$ in terms of the complementary unit Gaussian CDF function $Q(x)$, and use this to write $P\{|\mathcal{X}| > \alpha, |\mathcal{Y}| > \alpha\}$ in terms of $Q(x)$. (Remember commas mean intersections).
   (e) On your sketch of part (b), show the region over which you must integrate the joint pdf to find $P\{|\mathcal{X}| > \alpha, |\mathcal{Y}| > \alpha\}$. Use your sketch to prove the following result: $P\{|\mathcal{X}| > \alpha, |\mathcal{Y}| > \alpha\} < P\{\mathcal{X}^2 + \mathcal{Y}^2 > 2\alpha^2\}$ for $\alpha > 0$.
   (f) Show that inequality of part (e) implies that $Q(x) < \frac{1}{2}\exp(-x^2/2)$ for $x > 0$.

(g) On your sketch of parts (b) and (d), show the region over which you must integrate to find $P\{|\mathcal{X}| < \alpha, |\mathcal{Y}| < \alpha\}$, and prove that

$$P\{\mathcal{X}^2 + \mathcal{Y}^2 \leq \alpha^2\} < P\{|\mathcal{X}| < \alpha, |\mathcal{Y}| < \alpha\} < P\{\mathcal{X}^2 + \mathcal{Y}^2 < 2\alpha^2\}.$$

Use these inequalities to deduce the *lower* bound $Q(x) > \frac{1}{4}\exp(-x^2)$ for $x > 0$. Note that at $x = 0$, equality holds in the upper bound of part (f) but not in this lower bound .

4. The joint pdf of $\mathcal{X}$ and $\mathcal{Y}$ is given by $f_{\mathcal{X},\mathcal{Y}}(u,v) = \begin{cases} 2u, & 0 < u < 1, 0 < v < 1, \\ 0, & \text{elsewhere.} \end{cases}$

   Find the pdf of $\mathcal{Z} = \mathcal{X}^2\mathcal{Y}$.

5. Consider the random point $(\mathcal{X}, \mathcal{Y})$ whose joint pdf was specified in Problem 1.

   (a) Find $\mathsf{E}[\mathcal{X}]$ and $\mathsf{var}(\mathcal{X})$.

   (b) Explain why the random variable $\mathcal{Y}$ has the same mean and variance as $\mathcal{X}$.

   (c) Compute $\mathsf{E}[\mathcal{X}\mathcal{Y}]$ and hence determine $\mathsf{cov}(\mathcal{X}, \mathcal{Y})$.

6. Let $\mathsf{E}[\mathcal{X}] = 1, \mathsf{E}[\mathcal{Y}] = 4, \mathsf{var}(\mathcal{X}) = 4, \mathsf{var}(\mathcal{Y}) = 9$, and $\rho_{\mathcal{X},\mathcal{Y}} = 0.1$.

   (a) If $\mathcal{Z} = 2(\mathcal{X} + \mathcal{Y})(\mathcal{X} - \mathcal{Y})$, what is $\mathsf{E}[\mathcal{Z}]$?

   (b) If $\mathcal{T} = 2\mathcal{X} + \mathcal{Y}$ and $\mathcal{U} = 2\mathcal{X} - \mathcal{Y}$, what is $\mathsf{cov}(\mathcal{T}, \mathcal{U})$?

   (c) Find the mean and variance of $\mathcal{W} = 3\mathcal{X} + \mathcal{Y} + 2$.

   (d) If $\mathcal{X}$ and $\mathcal{Y}$ are jointly Gaussian random variables, and $\mathcal{W}$ is as defined in part (c), what is $P\{\mathcal{W} > 0\}$?

7. This problem has three independent parts. Do not apply the numbers from one part to the others.

   (a) If $\mathsf{var}(\mathcal{X} + \mathcal{Y}) = 36$ and $\mathsf{var}(\mathcal{X} - \mathcal{Y}) = 64$, what is $\mathsf{cov}(\mathcal{X}, \mathcal{Y})$? If you are also told that $\mathsf{var}(\mathcal{X}) = 3 \cdot \mathsf{var}(\mathcal{Y})$, what is $\rho_{\mathcal{X},\mathcal{Y}}$?

   (b) If $\mathsf{var}(\mathcal{X} + \mathcal{Y}) = \mathsf{var}(\mathcal{X} - \mathcal{Y})$, are $\mathcal{X}$ and $\mathcal{Y}$ uncorrelated ?

   (c) If $\mathsf{var}(\mathcal{X}) = \mathsf{var}(Y)$, are $\mathcal{X}$ and $\mathcal{Y}$ uncorrelated ?

There was a young man from Japan
Whose limericks never would scan
When asked why it was so
He replied "Ah so"
It is because I try and get as many words into the last line as ever I possibly can"