SOLUTIONS TO Final — ECE 313 – FALL 2007

1. (i) T, (ii) T, (iii) F, (iv) F, (v) F, (vi) T, (vii) F, (viii) T, (ix) T, (x) F.

Brief reasons:

(i) X and Y are independent.

(ii) X and Y are uncorrelated.

(iii) X and Y might not have the same variance.

(iv) W and Z are uncorrelated but might not be independent.

(v) Same as (iv)

(vi) X and Y are independent.

(vii) W and Z might not be independent.

(viii) $\text{cov}(2W, Z - W) = \text{cov}(2W, Z) - \text{cov}(2W, W) = -2\text{Var}(W)$

(ix) X and Y are independent.

(x) $E[X^2] \neq E[X]^2$.

2. (i) T, (ii) T, (iii) F, (iv) T, (v) F, (vi) F, (vii) T, (viii) T

Brief reasons:

(ii) $P(Y = 0) = 0.5$ and therefore Y is mixed.

(v) $\rho(X, Y) = -1/2$.

(vi) $\text{var}(X - Y) = \text{var}(X) + \text{var}(Y)$.

3. (i) b, (ii) b, (iii) a, (iv) a, (v) b, (vi) a, (vii) b, (viii) a, (xi) a, (x) c.

Brief reasons:

(ii) By symmetry

(iv) $E[(X - 1/2)Y] = E[XY] = E\big[E[XY|X = x]\big] = 0$.

(v) By symmetry

(vii) $f_{Z,W}(z, w) = 1,\ 0 < z \leq 1, 0 < w \leq 1$ and zero otherwise.

(viii) $E[X|Y] = 1/2$ so variance is 0.

(x) $\text{Var}(U) = \text{Var}(Z) = \text{Var}\big(\text{Unf}[0, 1]\big) = 1/12$.

4. (i) $P(Y = -1) = P(Y = 1) = 1/2$.

(ii) $E[Y] = 0$, $\text{var}(Y) = 1 \times 1/2 + 1 \times 1/2 = 1$

(iii) $\text{cov}(X, Y) = E[XY] = E\big[X\, E[Y|X]\big] = E\big[X\, sgn(X)\big] = E\big[|X|\big] = 2\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} x\, dx = \sqrt{\frac{2}{\pi}}$

(iv) $\hat{X}_{LMMSE} = \frac{\text{cov}(X,Y)}{\text{var}(Y)} Y = \sqrt{\frac{2}{\pi}}\, Y$ and $MSE = \text{var}(X) - \frac{\text{cov}(X,Y)}{\text{var}(Y)} = 1 - \frac{2}{\pi}$

(v) $f_{X|Y}(x|1) = 2f_X(x)$ for $x > 0$ and 0 otherwise.
Similarly, $f_{X|Y}(x|-1) = 2f_X(x)$ for $x < 0$ and 0 otherwise.

(vi) $\hat{X}_{MMSE} = E[X|Y] = \sqrt{\frac{2}{\pi}}Y$

5. (i)

$$f_{Y_1,Y_2|X}(y_1, y_2|x) = \frac{1}{2\pi}e^{-\frac{(y_1-x)^2+(y_2-2x)^2}{2}}$$

(ii)

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{Y_1,Y_2|X}(y_1, y_2|-1)P(X=-1) + f_{Y_1,Y_2|X}(y_1, y_2|1)P(X=1) \\ &= \frac{1}{4\pi}\left[e^{-\frac{(y_1+1)^2+(y_2+2)^2}{2}} + e^{-\frac{(y_1-1)^2+(y_2-2)^2}{2}}\right] \end{aligned}$$

(iii)

$$b_1Y_1 + b_2Y_2|X = 1 \sim \mathcal{N}(b_1 + 2b_2, b_1^2 + b_2^2)$$
$$b_1Y_1 + b_2Y_2|X = -1 \sim \mathcal{N}(-b_1 - 2b_2, b_1^2 + b_2^2)$$

(iv)

$$\begin{aligned} P_e &= \frac{1}{2}P\big(\mathcal{N}(-b_1 - 2b_2, b_1^2 + b_2^2) > 0\big) + \frac{1}{2}P\big(\mathcal{N}(b_1 + 2b_2, b_1^2 + b_2^2) < 0\big) \\ &= Q(\frac{b_1 + 2b_2}{\sqrt{b_1^2 + b_2^2}}) \end{aligned}$$

(v) To minimize P_e, we want to find (b_1, b_2) to maximize $\frac{(b_1+2b_2)^2}{b_1^2+b_2^2}$. The optimal $(b_1, b_2) = (1, 2)$.

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