**University of Illinois** **ECE 313**
**at Urbana-Champaign** **Fall 2007**

# Problem Set # 12

**Assigned:** Wednesday, November 28

**Due:** FRIDAY, December 7

**Coverage:** This homework set covers the topics of functions of many random variables (RVs); multiple functions of multiple RVs; jointly Gaussian RVs; expectation, covariance and correlation; conditional expectation; and mean-square estimation. The first topic was already partially covered by the previous homework set, but here you'll find a few more problems on it. The relevant sections of Ross for this problem set are: Chapter 6, Sections 6.3-6.7 (pp. 280-308), and Chapter 7, Sections 7.1-7.6 (pp. 327-387) and Section 7.8 (pp. 399-403). From the Lecture Notes on the class web-site, the relevant slides are Lectures 35-40.

In addition to the eight problems below (to be turned in), you are encouraged to attempt problems relevant to the topics above in the *Self-Test Problems and Exercises* section of Chapter 6 of Ross (pages 323-326), particularly 7, 10, 13, and 14, and that of Chapter 7 of Ross (pages 426-429), particularly 20, 22, 23 and 25; their solutions can be found in Appendix B of Ross.

**Note:** This is the last homework set of the semester. It is due on Friday (and not Wednesday) , December 7, which is the last day of instruction.

## PROBLEMS

**72.** Two jointly continuous random variables, $X$ and $Y$, have joint *pdf*

$$f_{X,Y}(x,y) = \begin{cases} 1/2, & 0 \le x < 1, 0 \le y < 1, \text{ and } 0 \le x+y < 1 \\ 3/2, & 0 \le x < 1, 0 \le y < 1, \text{ and } 1 \le x+y < 2 \\ 0, & \text{otherwise} \end{cases}$$

**(a)** Find $f_X(x)$, the marginal *pdf* of $X$.

**(b)** Compute $P(X + Y \le 3/2)$.

**(c)** Compute $P(X^2 + Y^2 \ge 1)$.

**(d)** Compute the conditional expectation of $X$ given $Y = y$, i.e., $E[X|Y = y]$

**73.** Let two jointly continuous random variables, $X$ and $Y$, have the joint *pdf*

$$f_{X,Y}(x,y) = \begin{cases} 2x, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find the *pdf* of the random variable $Z = X^2 Y$.

**74.** $X$ and $Y$ are two random variables with

$$E[X] = 1,\ E[Y] = 4,\ \text{var}(X) = 4,\ \text{var}(Y) = 9, \text{ and } \rho_{X,Y} = 0.1$$

**(a)** If $Z = 2(X + Y)(X - Y)$, what is $E[Z]$?

**(b)** If $T = 2X + Y$ and $U = 2X - Y$, what is $\text{cov}(T, U)$?

**(c)** If $W = 3X + Y + 2$, find $E[W]$ and $\text{var}(W)$.

**(c)** If $X$ and $Y$ defined above are jointly Gaussian, and $W$ is as defined in (c), what is $P(W > 0)$ ?

**75.** This problem has three independent parts, and thus can (should) be solved independently of each other.

**(a)** If $\text{var}(X + Y) = 36$ and $\text{var}(X - Y) = 64$, what is $\text{cov}(X, Y)$ ? If you are also told that $\text{var}(X) = 3\text{var}(Y)$, what is $\rho_{X,Y}$ ?

**(b)** If $\text{var}(X + Y) = \text{var}(X - Y)$, are $X$ and $Y$ uncorrelated ?

**(c)** If $\text{var}(X) = \text{var}(Y)$, are $X$ and $Y$ uncorrelated ?

**76.** $X$ and $Y$ are two independent and identically distributed uniform random variables on $(0, 1)$.

**(a)** Find the joint *pdf*s of: *(i)* $U = X + Y$ and $V = X/Y$; *(ii)* $U = X$ and $V = X/Y$; *(iii)* $U = X + Y$ and $V = X/(X + Y)$.

**(b)** Use the result in *(a(i))* above to obtain the *pdf* of $U = X + Y$.

**(c)** Now use the result in *(a(iii))* above to obtain the *pdf* of $U = X + Y$.

**77.** If $X$, $Y$ and $Z$ are independent and identically distributed exponential random variables with parameter $\lambda = 1$, find the joint *pdf* of the three random variables $U$, $V$, and $W$, given by

$$U = X + Y,\ V = X + Z,\ W = Y + Z\,.$$

Note: For Jacobian of functions of more than two variables, see Ross, page 305.

**78.** $X$ and $Y$ are jointly continuous random variables with joint *pdf*

$$f_{X,Y}(x, y) = e^{-(x/y)} e^{-y}/y\,,\ \ 0 < x < \infty,\ 0 < y < \infty\,.$$

Find the function g(Y) that minimizes the quantity $E[(X^3 - g(Y))^2]$.

**79.** This problem has two parts which can be solved independently, or you can use the solution of (a) in obtaining the solution to (b).

**(a)** Given three random variables $X, Y, Z$, the *best affine predictor* of $X$ with respect to $Y$ and $Z$ is $a + bY + cZ$, where $a, b, c$ are chosen to minimize

$$E[(X - (a + bY + cZ))^2]\,.$$

Determine $a, b$, and $c$ in terms of covariances and variances involving the three random variables.

**(b)** Given two random variables $X, Y$, the *best quadratic predictor* of $X$ with respect to $Y$ is $a + bY + cY^2$, where $a, b, c$ are chosen to minimize

$$E[(X - (a + bY + cY^2))^2]\,.$$

Determine $a, b$, and $c$.