University of Illinois **ECE 313**
at Urbana-Champaign **Fall 2007**

Problem Set # 6

Assigned: Wednesday, September 26
Due: Wednesday, October 3
Coverage: This homework set covers the topics of "Conditional Probability," "Independent Events" and "Decision Making Under Uncertainty." The first topic was already covered by the previous homework set, but here you'll find a few more questions on it. Both the first and the second topics are treated in Chp 3 of Ross. For the third topic, supplementary notes are posted on the course web-site.

In addition to the seven problems below (to be turned in), you are encouraged to attempt problems in the *Self-Test Problems and Exercises* section of Chapter 3 of Ross (pages 128-131), particularly 11, 15, 17, 18 and 19; their solutions can be found in Appendix B of Ross.

Note: This is the last homework set before the first midterm exam, scheduled for October 8, 2007.

PROBLEMS

33. Suppose that we want to generate the outcome of the flip of a fair coin but that all we have at our disposal is a biased coin which lands on heads with some unknown probability p that need not be equal to $\frac{1}{2}$. Consider the following procedure for accomplishing our task.
1. Flip the coin.
2. Flip the coin again.
3. If both flips land heads or both land tails, return to step 1.
4. If the two flips land differently, then stop and declare the result of the last flip to be the result of the experiment.

(a) Show that the result is equally likely to be either heads or tails, regardless of what p is.
(b) Could we use a simpler procedure that continues to flip the coin until the last two flips are different and then lets the result be the outcome of the final flip?

34. Suppose that E and F are mutually exclusive events of an experiment. Show that if independent trials of this experiment are performed, then E will occur before F with probability $P(E)/[P(E)+P(F)]$.

35. Prove that if $E_1, E_2, \ldots, E_n$ are independent events, then

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = 1 - \prod_{i=1}^{n}[1 - P(E_i)]$$

36. Consider two independent tosses of a fair coin. Let A be the event that the first toss lands heads, B be the event that the second toss lands heads, and C be the event that both land on the same side. Show that A, B, C are *pairwise independent* (that is, A and B are independent, B and C are independent, and A and C are independent), but they are *not* independent.

37. Let E_1, E_2, F be three events on a sample space Ω. We say that E_1 and E_2 are "independent conditioned on F" if $P(E_1E_2|F) = P(E_1|F)P(E_2|F)$. The goal of this problem is to show that *conditional independence* does not necessarily imply (unconditional) independence.

Let Ω be an urn with six balls, numbered 1 through 6, with each one being equally likely to be drawn. Let $E_1 = \{3, 4, 5\}$, $E_2 = \{2, 4, 6\}$, $F = \{1, 2, 3, 4\}$

E_1 here could correspond, for example, to the event that a ball drawn is one of the three carrying numbers 3, 4 or 5; a similar interpretation would apply to E_2 and F.

(a) Show that events E_1 and E_2 are independent when conditioned on F.

(b) Show that E_1 and E_2 are not (unconditionally) independent.

(c) Can you come up with a similar set-up, but with fewer balls (for example, 5 or 4) and construct events E_1, E_2, F, where E_1 and E_2 are independent when conditioned on F, but are not unconditionally independent?

38. In a factory there are two production lines manufacturing a certain electrical component. The first line A produces a defective component with probability $p_0 = 0.05$, whereas the second line B is slightly inferior and produces a defective component with probability $p_1 = 0.10$. When the factory fulfills orders for its customers, it always specifies whether the batch was produced at line A or B. Suppose that there is an unlabeled batch consisting of $N = 100$ components. An inspection reveals that there are 7 defective components.

(a) What are the likelihoods that this batch was produced at line A or B?

(b) What is the maximum likelihood (ML) decision?

(c) Now obtain the ML decision rule in terms of the batch size N and the number of defective components k.

39. Suppose you have 10 coins. You know that while 8 of these coins are fair, the remaining 2 are biased, and for the biased coins the probability of getting a head is 0.7. However, you cannot tell them apart. So you randomly pick a coin, flip it 5 times, and count the number of heads in order to decide whether the coin is biased or not. [As a convention, let H_0 denote the hypothesis corresponding to the fair coin, and H_1 be the alternative hypothesis.]

(a) Construct the likelihood matrix, state the maximum likelihood (ML) decision rule, and find the probability of making a wrong decision.

(b) Construct the joint probability matrix, state the maximum a posteriori (MAP) decision rule, and find the probability of making the wrong decision under the MAP rule.

(c) Now suppose that there is some cost associated with making wrong decisions, and a reward associated with making correct decisions. As in the *Notes*, let C_{ij} denote the cost of deciding hypothesis H_i is true when actually H_j is true, with $i, j = 0, 1$ (where C_{00} and C_{11} are negative, since they correspond to rewards). Obtain the *Bayes' minimum average cost* decision rule when $C_{00} = C_{11} = -1, C_{01} = 2, C_{10} = 1$.