University of Illinois, Urbana–Champaign Fall 2007

ECE 313: Solutions to Homework 10

60. (a) According to the law of total probability, we have

$$\int_0^1 c(1-x)dx = cx - \frac{cx^2}{2}\Big|_0^1 = \frac{c}{2} = 1$$

Then, we get $c = 2$.

(b)

$$F_Y(b) = P(Y \le b) = P((1-X)^2 \le b) = P(1-\sqrt{b} \le X \le 1) \text{for } 0 \le b \le 1$$
$$= \begin{cases} \int_{1-\sqrt{b}}^1 2(1-x)dx = b & b \ge 0 \\ 0 & b < 0 \end{cases}$$

(c) Take a derivative of $F_Y(b)$, we obtain the pdf, whibh is

$$f_Y(b) = F_Y'(b) = \begin{cases} 1 & 0 \le b \le 1 \\ 0 & otherwise \end{cases}$$

Y is then uniformly distributed over $[0, 1]$.

61. (a)

$$E[X] = \int_0^\infty \frac{x^2}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}} dx = \int_0^\infty -x d e^{\frac{-x^2}{2\sigma^2}} = -xe^{\frac{-x^2}{2\sigma^2}}\Big|_0^\infty + \int_0^\infty e^{\frac{-x^2}{2\sigma^2}} dx = \int_0^\infty e^{\frac{-x^2}{2\sigma^2}} dx$$

We know from Gaussian $(0, \sigma^2)$ distribution that $\int_{-\infty}^\infty e^{\frac{-x^2}{2\sigma^2}} = \sqrt{2\pi}\sigma$. The symmetry property gives that $E[X] = \int_0^\infty e^{\frac{-x^2}{2\sigma^2}} dx = \sqrt{\frac{\pi}{2}}\sigma$.

(b) Then, $Y = \min(X, E[X]) = \min(X, \sqrt{\frac{\pi}{2}}\sigma)$.

$$F_Y(c) = P(Y \le c) = \begin{cases} 0 & c < 0 \\ 1 & c \ge \sqrt{\frac{\pi}{2}}\sigma \\ \int_0^c \frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}} dx = -e^{\frac{-x^2}{2\sigma^2}}\Big|_0^c = 1 - e^{\frac{-c^2}{2\sigma^2}} & otherwise \end{cases}$$

Y is a mixed random variable since $F_Y(\sqrt{\frac{\pi}{2}}\sigma^-) = 1 - e^{-\pi/4} \ne F_Y(\sqrt{\frac{\pi}{2}}\sigma^+) = 1$.

(c)

$$E[Y] = \int_0^{\sqrt{\frac{\pi}{2}}\sigma} (1 - F_Y(c))dc = \int_0^{\sqrt{\frac{\pi}{2}}\sigma} e^{\frac{-c^2}{2\sigma^2}} dc$$
$$= \sqrt{2\pi}\sigma \int_0^{\sqrt{\frac{\pi}{2}}} \frac{1}{\sqrt{2\pi}} e^{\frac{c^2}{2}} dc$$
$$= \sqrt{2\pi}\sigma(\Phi(\sqrt{\frac{\pi}{2}}) - \frac{1}{2}) \approx 0.99\sigma$$

62. Let $X \sim \mathsf{Binom}(150, 0.05)$ be the number of unacceptable cell phones. Using the DeMoivre-Laplace limit theorem to approximate Binomial distribution by Gaussian, we have $\mu = np = 150 \times 0.05 = 7.5, \sigma = \sqrt{np(1-p)} = 2.6693$. Then the approximated probability with continuity correction that at most 10 cell phones will not be acceptable is

$$P(-0.5 \le X \le 10.5) = P(\frac{-0.5-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{10.5-\mu}{\sigma}) = \Phi(1.1239) - \Phi(-2.9970) = 0.8681$$

63. From the hazard rate function, we obtain the CDF for the age of a male smoker, X, is

$$
\begin{aligned}
F_X(t) &= 1 - exp\{-\int_{40}^{t} \lambda(u)du\} \\
&= 1 - exp\{-\int_{40}^{t} 0.027 + 0.00025(u-40)^2 du\} \\
&= 1 - exp\{-0.000083t^3 + 0.01t^2 - 0.427t - 6.392\}
\end{aligned}
$$

(a) The probability that a 40 years old male smoker lives over age 50 is

$$P(X > 50 | X > 40) = \frac{1 - F_X(50)}{1 - F_X(40)} = 0.71688$$

(b) Similarly, we have

$$P(X > 60 | X > 40) = \frac{1 - F_X(60)}{1 - F_X(40)} = 0.3147$$

64. The CDF of X given its hazard rate $\lambda(t)$ is $F_X(t) = 1 - exp\{-\int_0^x \lambda(t)dt\}$. The CDF of $Y = aX$ is

$$
\begin{aligned}
F_Y(t) = P(Y \le t) = P(aX \le t) &= P(X \le \frac{t}{a}) \\
&= 1 - exp\{-\int_0^{\frac{t}{a}} \lambda(u)du\} \\
&= 1 - exp\{-\int_0^{t} \lambda(\frac{u}{a})d\frac{u}{a}\} \\
&= 1 - exp\{-\int_0^{t} \frac{1}{a}\lambda(\frac{u}{a})du\}
\end{aligned}
$$

Therefore, the hazard rate for Y is $\frac{1}{a}\lambda(\frac{t}{a})$. To verify the relationship we compute the hazard rate for X and Y of uniform distribution. For $X \sim \mathsf{Unif}(0,1)$, the CDF is

$$F_X(c) = \begin{cases} 0 & c < 0 \\ c & 0 \le c < 1 \\ 1 & c \ge 1 \end{cases}$$

The pdf is

$$f_X(c) = \begin{cases} 1 & 0 \le c < 1 \\ 0 & otherwise \end{cases}$$

Then, the hazard rate could be computed by $\lambda(t) = \frac{f_X(t)}{1-F_X(t)} = \frac{1}{1-t}$. For $Y = aX$, the CDF is

$$F_Y(c) = \begin{cases} 0 & c < 0 \\ \frac{c}{a} & 0 \le c < a \\ 1 & c \ge a \end{cases}$$

The pdf is

$$f_Y(c) = \begin{cases} 0 & otherwise \\ \frac{1}{a} & 0 \le c < a \end{cases}$$

Then, the hazard rate could be computed by $\frac{f_Y(t)}{1-F_Y(t)} = \frac{1}{a}\frac{1}{1-\frac{t}{a}} = \frac{1}{a}\lambda(\frac{t}{a})$. Our solution is hereby verified.

65. (a)

$$\int_{-1}^{1} c_0 x^2 dx = \frac{cx^3}{3}\Big|_{-1}^{1} = \frac{2c}{3} = 1$$

$$\int_{-3}^{3} c_1(3-|x|)dx = 2\int_{0}^{3} c_1(3-x)dx = 9c_1 = 1$$

Then, we have $c_0 = \frac{3}{2}, c_1 = \frac{1}{9}$.

(b) Using the ML rule, decision H_1 is given for the region Γ_1 such that $f_1 > f_0$ and decision H_0 is given for the complementary region of Γ_1, Γ_0, such that $f_1 < f_0$. It is obvious that $[-3,-1) \cup (1,3] \in \Gamma_1$ since $f_0 = 0$ in those regions. Then we need to find the region that $f_1 > f_0$ in $[-1,1]$. By symmetry, we only need to find the region for $x > 0$.

$$\begin{aligned} f_1(x) &> f_0(x) \\ \frac{1}{9}(3-x) &> \frac{3}{2}x^2 \\ 27x^2 + 2x - 6 &< 0 \\ 0 < x &< \frac{\sqrt{163}-1}{27} = 0.4358 \end{aligned}$$

Therefore, we have $\Gamma_1 = [-3,-1) \cup (-0.4358, 0.4358) \cup (1,3]$ and $\Gamma_0 = [-1,-0.4358] \cup [0.4358, 1]$. The error probability is

$$\pi \int_{\Gamma_1} f_0(x)dx + (1-\pi)\int_{\Gamma_0} f_1(x)dx = 2(0.25\int_{0}^{0.4358} f_0(x)dx + 0.75\int_{0.4358}^{1} f_1(x)dx) = 0.2353$$

(c) Using the MAP rule, decision H_1 is given for the region Γ_1 such that $(1-\pi)f_1 > \pi f_0$ and decision H_0 is given for the complementary region of Γ_1, Γ_0, such that $(1-\pi)f_1 < \pi f_0$. It is the same that $[-3,-1) \cup (1,3] \in \Gamma_1$ since $f_0 = 0$ in those regions. Then we need to find the region that $(1-\pi)f_1 > \pi f_0$ in $[-1,1]$. By symmetry, we only need to find the region for $x > 0$.

$$\begin{aligned} (1-\pi)f_1(x) &> \pi f_0(x) \\ \frac{1}{12}(3-x) &> \frac{3}{8}x^2 \\ 9x^2 + 2x - 6 &< 0 \\ 0 < x &< \frac{\sqrt{55}-1}{9} = 0.7129 \end{aligned}$$

Therefore, we have $\Gamma_1 = [-3,-1) \cup (-0.7129, 0.7129) \cup (1,3]$ and $\Gamma_0 = [-1,-0.7129] \cup [0.7129, 1]$. The error probability is

$$\pi \int_{\Gamma_1} f_0(x)dx + (1-\pi)\int_{\Gamma_0} f_1(x)dx = 2(0.25\int_{0}^{0.7129} f_0(x)dx + 0.75\int_{0.7129}^{1} f_1(x)dx) = 0.1931$$

66. (a) The probability that the ith ball is white is $P(X_i = 1) = \frac{\#remaining\, white\, balls}{13-i}$ and

$P(X_i = 0) = 1 - P(X_i = 1)$. The pmf is thus given by,

$$\begin{aligned}
P(X_1 = 0, X_2 = 0) &= \frac{8}{13}\frac{7}{12} = \frac{14}{39} = 0.3590 \\
P(X_1 = 0, X_2 = 1) &= \frac{8}{13}\frac{5}{12} = \frac{10}{39} = 0.2564 \\
P(X_1 = 1, X_2 = 0) &= \frac{5}{13}\frac{8}{12} = \frac{10}{39} = 0.2564 \\
P(X_1 = 1, X_2 = 1) &= \frac{5}{13}\frac{4}{12} = \frac{5}{39} = 0.1282
\end{aligned}$$

To verify the answer, sum up $\frac{14+10+10+5}{39} = 1$ so it is a valid pmf.

(b) Similarly, the pmf for X_1, X_2, X_3 is

$$\begin{aligned}
P(X_1 = 0, X_2 = 0, X_3 = 0) &= \frac{8}{13}\frac{7}{12}\frac{6}{11} = \frac{28}{143} = 0.1958 \\
P(X_1 = 0, X_2 = 1, X_3 = 0) &= \frac{8}{13}\frac{5}{12}\frac{7}{11} = \frac{70}{429} = 0.1632 \\
P(X_1 = 1, X_2 = 0, X_3 = 0) &= \frac{5}{13}\frac{8}{12}\frac{7}{11} = \frac{70}{429} = 0.1632 \\
P(X_1 = 1, X_2 = 1, X_3 = 0) &= \frac{5}{13}\frac{4}{12}\frac{8}{11} = \frac{40}{429} = 0.0932 \\
P(X_1 = 0, X_2 = 0, X_3 = 1) &= \frac{8}{13}\frac{7}{12}\frac{5}{11} = \frac{70}{429} = 0.1632 \\
P(X_1 = 0, X_2 = 1, X_3 = 1) &= \frac{8}{13}\frac{5}{12}\frac{4}{11} = \frac{40}{429} = 0.0923 \\
P(X_1 = 1, X_2 = 0, X_3 = 1) &= \frac{5}{13}\frac{8}{12}\frac{7}{11} = \frac{40}{429} = 0.0923 \\
P(X_1 = 1, X_2 = 1, X_3 = 1) &= \frac{5}{13}\frac{4}{12}\frac{3}{11} = \frac{5}{143} = 0.0350
\end{aligned}$$

Again, one can verify the answer by summing them up to see if it equals one.

(c)

$$\begin{aligned}
\mathsf{E}[Y] &= \mathsf{E}[X_1 X_2] = 1\frac{5}{39} + 0(1 - \frac{5}{39}) = \frac{5}{39} = 0.1282 \\
\mathsf{E}[Z] &= \mathsf{E}[X_1 X_2 X_3] = \frac{5}{143} = 0.0350 \\
\mathsf{E}[Z^2] &= 0.0350 \\
\mathsf{var}[Z] &= \mathsf{E}[Z^2] - \mathsf{E}[Z]^2 = 0.0338
\end{aligned}$$

(d)

$$P(X_3 = 1 | X_1 = 1, X_2 = 0) = \frac{P(X_1 = 1, X_2 = 0, X_3 = 1)}{P(X_1 = 1, X_2 = 0)} = \frac{4}{11} = 0.3636$$

(e)

$$P(X_1 = 1, X_2 = 0 | X_3 = 1) = \frac{P(X_1 = 1, X_2 = 0, X_3 = 1)}{\sum_{X_1, X_2 \in \{0,1\}} P(X_1 = i, X_2 = j, X_3 = 1)} = \frac{8}{33} = 0.2424$$