67. $F_X(c) = 1 - e^{-\lambda_x c}$ for $c \geq 0$ and $F_Y(c) = 1 - e^{-\lambda_y c}$ for $c \geq 0$.

(a) X and Y are independent so the probability that the maximum of them is less than or equal to c equals to the product of the probability of each being $\leq c$. Therefore, for $c \geq 0$, we have,

$$\begin{aligned} F_{\max(X,Y)}(c) &= P(X \leq c, Y \leq c) = P(X \leq c)P(Y \leq c) = F_X(c)F_Y(c) \\ &= 1 - e^{-\lambda_x c} - e^{-\lambda_y c} + e^{-c(\lambda_x + \lambda_y)} \end{aligned}$$

The CDF is zero for $c < 0$. The pdf of $\max(X, Y)$ is

$$f_{\max(X,Y)} = F'_{\max(X,Y)} = \lambda_x e^{-\lambda_x c} + \lambda_y e^{-\lambda_y c} - (\lambda_x + \lambda_y)e^{-c(\lambda_x+\lambda_y)}, c \geq 0$$

The pdf is zero for $c < 0$. A similar procedure can be applied to compute the pdf of $\min(X, Y)$. For nonnegative c, we have,

$$F_{\min(X,Y)}(c) = P(X \leq c \cup Y \leq c) = P(X \leq c) + P(Y \leq c) - P(X \leq c, Y \leq c)$$

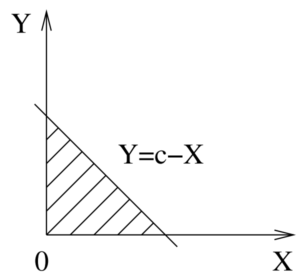
Using the result from the first part of the solution, we will get

$$F_{\min(X,Y)}(c) = 1 - e^{-c(\lambda_x+\lambda_y)}$$

In the end,

$$f_{\min(X,Y)}(c) = F'_{\min(X,Y)}(c) = (\lambda_x + \lambda_y)e^{-c(\lambda_x+\lambda_y)}, c \geq 0$$

(b) To find the pdf of $X+Y$, we need to sum the two dimensional probability over the shaded area.



For $c \geq 0$, we have,

$$\begin{aligned} F_{X+Y}(c) = P(X + Y \leq c) &= \int_0^c \int_0^{c-x} f_{X,Y}(x, y)dydx = \int_0^c \int_0^{c-x} f_X(x)f_Y(y)dydx \\ &= \int_0^c \int_0^{c-x} \lambda_x e^{-\lambda_x x}\lambda_y e^{-\lambda_y y}dydx \\ &= \begin{cases} 1 - \frac{\lambda_x e^{-c\lambda_y} - \lambda_y e^{-c\lambda_x}}{\lambda_x - \lambda_y} & \lambda_x \neq \lambda_y \\ 1 - (1 + c\lambda)e^{-c\lambda} & \lambda_x = \lambda_y = \lambda \end{cases} \end{aligned}$$

Then, for $c \geq 0$, the pdf is

$$\begin{aligned} f_{X+Y}(c) &= F'_{X+Y}(c) \\ &= \begin{cases} \frac{\lambda_x\lambda_y}{\lambda_x-\lambda_y}(e^{-c\lambda_y} - e^{-c\lambda_x}) & \lambda_x \neq \lambda_y \\ c\lambda^2 e^{-c\lambda} & \lambda_x = \lambda_y = \lambda \end{cases} \end{aligned}$$

(c) For $c \geq 0$,

$$f_{X|X+Y}(c|a) = \frac{f_{X,X+Y}(c,a)}{f_{X+Y}(a)} = \frac{f_{X,Y}(c,a-c)}{f_{X+Y}(a)}$$
$$= \begin{cases} \frac{(\lambda_x - \lambda_y)e^{-\lambda_x c - \lambda_y a + \lambda_y c}}{e^{-a\lambda_y} - e^{-a\lambda_x}} & \lambda_x \neq \lambda_y \\ \frac{1}{a} & \lambda_x = \lambda_y = \lambda \end{cases}$$

(d) For $b \geq 0$,

$$F_{X|X}(c|X \geq b) = P(X \leq c|X \geq b) = \begin{cases} \frac{P(b \leq X \leq c)}{P(X \geq b)} = 1 - e^{-\lambda_x(c-b)} & b \leq c \\ 0 & b > c \end{cases}$$

For $b < 0$,

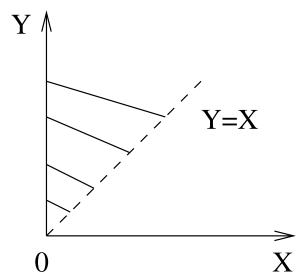
$$F_{X|X}(c|X \geq b) = \begin{cases} \frac{P(0 \leq X \leq c)}{P(X \geq 0)} = 1 - e^{-\lambda_x c} & c > 0 \\ 0 & o/w \end{cases}$$

Thus, the pdf is

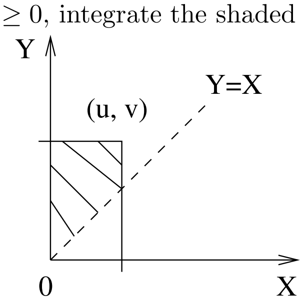
$$f_{X|X}(c|X \geq b) = \begin{cases} \lambda_x e^{-\lambda_x(c-b)} & 0 \leq b \leq c \\ \lambda_x e^{-\lambda_x c} & b < 0, c > 0 \\ 0 & o/w \end{cases}$$

68. The answer is no. If $F(x,y)$ is a valid joint CDF, we can find a valid marginal CDF of X from the joint CDF, $F_X(x) = F(x,\infty) = 1$ for all x. Since a valid CDF requires $F(-\infty) = 0$, $F_X(x)$ is not a valid CDF. Therefore, the joint CDF is not valid.

69. (a) The nonzero region is indicated by shaded area.



(b) For any point (u,v), where $u \geq 0, v \geq 0$, integrate the shaded area to get the CDF.



$$F_{X,Y}(u,v) = P(X \leq u, Y \leq v)$$
$$= \begin{cases} \int_0^v \int_0^y 2e^{-(x+y)}dxdy & 0 \leq v \leq u \\ \int_0^u \int_0^y 2e^{-(x+y)}dxdy + \int_0^u \int_u^v 2e^{-(x+y)}dxdy & 0 \leq u < v \end{cases}$$
$$= \begin{cases} e^{-2v} - 2e^{-v} + 1 & 0 \leq v \leq u \\ 1 + 2e^{-v-u} - e^{-2u} - 2e^{-v} & 0 \leq u < v \end{cases}$$

(c) Compute marginal CDF directly from $F_{X,Y}$,

$$F_X(u) = F_{X,Y}(u,+\infty) = \begin{cases} 1-e^{-2u} & u \geq 0 \\ 0 & o/w \end{cases}$$

$$F_Y(v) = F_{X,Y}(+\infty,v) = \begin{cases} e^{-2v}-2e^{-v}+1 & v \geq 0 \\ 0 & o/w \end{cases}$$

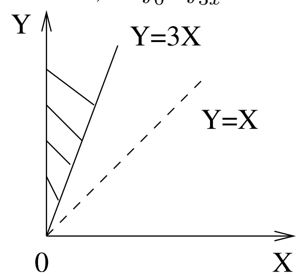
(d) Find marginal pdf by integrating $f_{X,Y}$,

$$f_X(u) = \int_u^\infty f_{X,Y}(u,y)dy = \begin{cases} 2e^{-2u} & u \geq 0 \\ 0 & o/w \end{cases}$$

$$f_Y(v) = \int_0^v f_{X,Y}(x,v)dx = \begin{cases} 2e^{-v}-2e^{-2v} & v \geq 0 \\ 0 & o/w \end{cases}$$

Differentiating the marginal CDF to get the same answer.

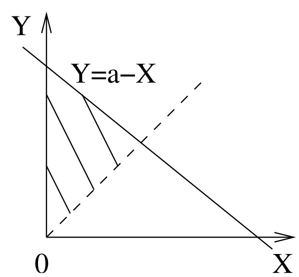
(e) Integrate the shaded area to find $P(Y > 3X) = \int_0^\infty \int_{3x}^\infty 2e^{-(x+y)}dydx = \frac{1}{2}$.



(f) Integrate the shaded area to get the probability that

$$P(X+Y \leq \alpha) = \int_0^{\alpha/2} \int_x^{\alpha-x} 2e^{-(x+y)}dxdy = 1-(1+\alpha)e^{-\alpha}$$

where $\alpha \geq 0$.



(g) The CDF of $Z = X+Y$ equals,

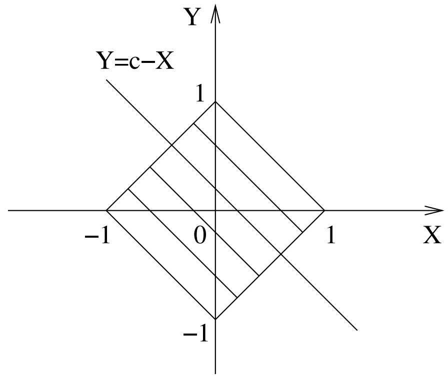
$$F_Z(c) = P(X+Y \leq c) = \begin{cases} 1-(1+c)e^{-c} & c \geq 0 \\ 0 & o/w \end{cases}$$

And we can get the pdf,

$$f_Z(c) = \begin{cases} ce^{-c} & c \geq 0 \\ 0 & o/w \end{cases}$$

70. Since x, y, z in the probability density function g is interchangeable, the probability mass should be symmetric over the plane $X = Y$, plane $Y = Z$, and plane $Z = X$. Three planes join at the line $X = Y = Z$ and divides the space into six equal parts. The region of $X < Y < Z$ is exactly one of them. The probability mass in any part of the six is symmetric to the other parts. Because the total probability mass has to be 1, the portion of $P(X < Y < Z)$ shall equal to 1/6.

71. The joint random variables have uniform probability mass in the following shaded area.



The total probability in that region should sum up to one and the shaded region is of size 2, so we obtain the density function,

$$f_{X,Y}(u,v) = \begin{cases} \frac{1}{2} & c \in area \\ 0 & o/w \end{cases}$$

(a) The marginal pdf of X is calculated by integrating the two dimensional probability mass along the y axis and is given by,

$$f_X(x) = \begin{cases} \int_{|x|-1}^{1-|x|} \frac{1}{2} = 1 - |x| & |x| \leq 1 \\ 0 & o/w \end{cases}$$

By symmetry, we have the pdf of Y,

$$f_Y(y) = \begin{cases} 1 - |y| & |y| \leq 1 \\ 0 & o/w \end{cases}$$

Since $f_X(x)f_Y(y) \neq f_{X,Y}(x,y)$ at points other than the corner points, they are not independent. Actually, the shape of nonzero mass has to be a square (no rotation) if X, Y are independent.

(b) Again by symmetry, $\mathsf{E}[X] = 0$. $\mathsf{var}(X) = \mathsf{E}[X^2] = 1/6$.

(c) The CDF of $Z = X+Y$ is computed by summing the shaded area below the line $Y = c-X$ for some $|c| \leq 1$,

$$F_Z(c) = P(X + Y \leq c) = \begin{cases} \frac{1+c}{2} & |c| \leq 1 \\ 1 & c > 1 \\ 0 & o/w \end{cases}$$

The pdf becomes,

$$f_Z(c) = \begin{cases} \frac{1}{2} & |c| \leq 1 \\ 0 & o/w \end{cases}$$