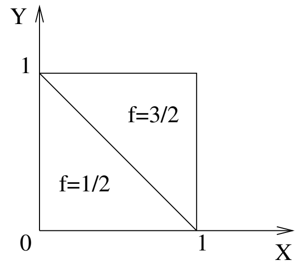
72. The joint probability density function is illustrated in the following figure.

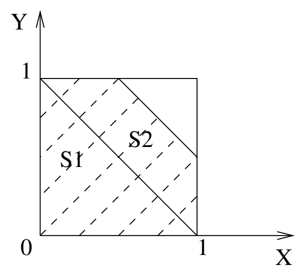


(a) For $0 \leq x \leq 1$, integrating the density function along the y-axis, we have

$$f_X(x) = \int_0^{1-x} \frac{1}{2} + \int_{1-x}^{1} \frac{3}{2} = \frac{1}{2} + x$$

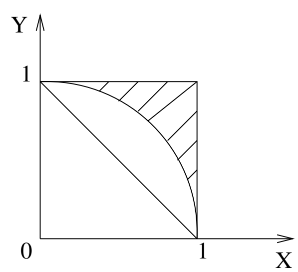
Otherwise, $f_X(x) = 0$.

(b) The probability is computed by integrating the joint density function over following indicated region S_1, S_2. Since the density is uniformly $\frac{1}{2}$ in S_1 and $\frac{3}{2}$ in S_2, we just simply multiply the area by the density.



$$P(X + Y \leq 3/2) = \frac{1}{2}S_1 + \frac{3}{2}S_2 = \frac{1}{4} + \frac{3}{2}\frac{3}{8} = \frac{13}{16}$$

(c) In the same way, the area to integrate is the remaining part of the square cut by an quarter arc.



$$P(X^2 + Y^2 \geq 1) = \frac{3}{2}(1 - \frac{\pi}{4}) = 0.3219$$

(d) First, we need to find the conditional probability density function for $0 \leq y \leq 1$. By symmetry, the marginal pdf of $f_Y(y) = \frac{1}{2} + y, 0 \le y \le 1$.

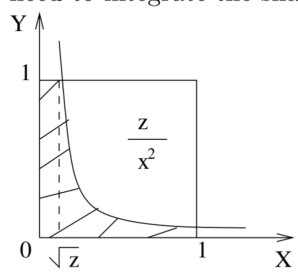
$$
\begin{aligned}
f_{X|Y=y}(u|v) &= \frac{f_{X,Y}(u,y)}{f_Y(y)} \\
&= \begin{cases} \frac{\frac{1}{2}}{\frac{1}{2}+y} & 0 \le u \le 1-y \\ \frac{\frac{3}{2}}{\frac{1}{2}+y} & 1-y < u \le 1 \end{cases}
\end{aligned}
$$

Then, the conditional expectation is computed by

$$
\mathsf{E}[X|Y=y] = \int_0^{1-y} u \frac{1}{1+2y} du + \int_{1-y}^{1} u \frac{3}{1+2y} du = \frac{1+4y-2y^2}{2+4y}
$$

for $0 \le y \le 1$. Otherwise, $\mathsf{E}[X|Y=y] = 0$.

73. To obtain the CDF of $Z = X^2 Y$, we need to integrate the shaded area for any $0 < z < 1$,



$$
F_Z(z) = \int_0^{\sqrt{z}} \int_0^1 2x dy dx + \int_{\sqrt{z}}^{1} \int_0^{\frac{z}{x^2}} 2x dy dx = z - z \ln z
$$

Differentiate the CDF to get the pdf of Z,

$$
f_Z(z) = -\ln z
$$

for $0 < z < 1$. Otherwise $f_Z(z) = 0$.

74. (a) $\mathsf{E}[Z] = 2\mathsf{E}[X^2] - 2\mathsf{E}[Y^2] = 2(\mathsf{var}(X) + (\mathsf{E}[X])^2 - \mathsf{var}(Y) - (\mathsf{E}[Y])^2) = -40$

(b) $\mathsf{cov}(T,U) = \mathsf{cov}(2X+Y, 2X-Y) = \mathsf{E}[(2X+Y)(2X-Y)] - \mathsf{E}[2X+Y]\mathsf{E}[2X-Y] = 4\mathsf{E}[X^2] - \mathsf{E}[Y^2] - 4(\mathsf{E}[X])^2 + (\mathsf{E}[Y])^2 = 7$

(c)

$$
\begin{aligned}
\mathsf{cov}(X,Y) &= \rho_{X,Y}\sqrt{\mathsf{var}(X)\mathsf{var}(Y)} = 0.6 \\
\mathsf{E}[W] &= \mathsf{E}[3X+Y+2] = 3\mathsf{E}[X] + \mathsf{E}[Y] + 2 = 9 \\
\mathsf{var}(W) &= \mathsf{var}(3X+Y) = 9\mathsf{var}(X) + \mathsf{var}(Y) + 6\mathsf{cov}(X,Y) = 48.6
\end{aligned}
$$

(d) Linear combination of joint Gaussian r.v. is also a Gaussian r.v. So W is Gaussian with mean 9 and variance 46.8. $\Pr(W > 0) = \Pr(\frac{W-9}{\sqrt{46.8}} > \frac{-9}{\sqrt{46.8}}) = \Phi(1.3156) = 0.9058$.

75. (a)

$$
\begin{aligned}
\mathsf{var}(X+Y) &= \mathsf{var}(X) + \mathsf{var}(Y) + 2\mathsf{cov}(X,Y) = 36 \\
\mathsf{var}(X-Y) &= \mathsf{var}(X) + \mathsf{var}(Y) - 2\mathsf{cov}(X,Y) = 64
\end{aligned}
$$

Solving for $\mathsf{cov}(X,Y)$, we can get $\mathsf{cov}(X,Y) = -7$. With $\mathsf{var}(X) = 3\mathsf{var}(Y)$, further we have $\mathsf{var}(X) = 37.5, \mathsf{var}(Y) = 12.5$.

$$
\rho_{X,Y} = \frac{\mathsf{cov}(X,Y)}{\sqrt{\mathsf{var}(X)\mathsf{var}(Y)}} = -0.3233
$$

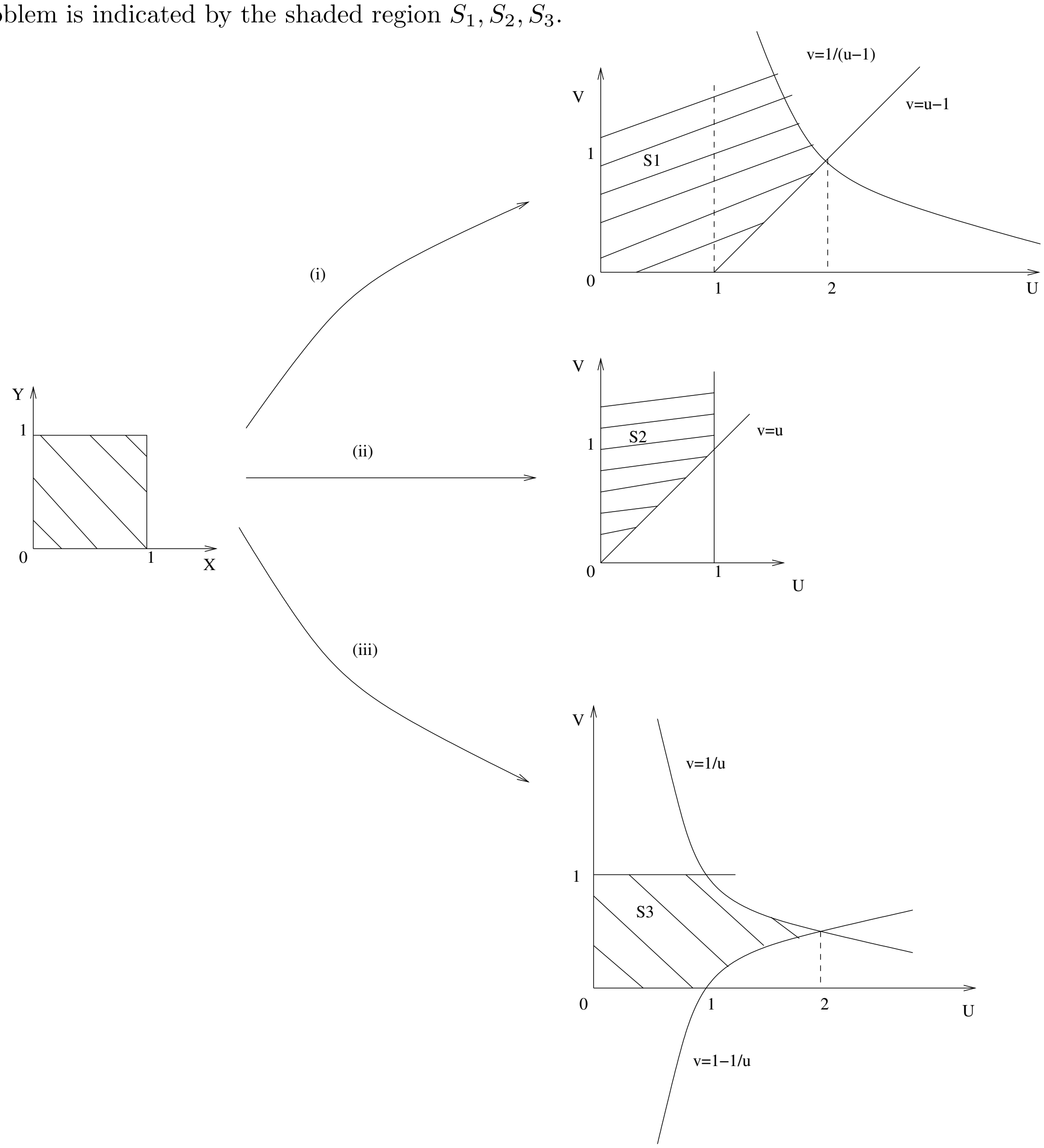
(b) If $\mathsf{var}(X+Y) = \mathsf{var}(X-Y)$, we can get $\mathsf{cov}(X,Y) = 0$. Thus X, Y are uncorrelated.

(c) $\mathsf{var}(X), \mathsf{var}(Y)$ alone do not give us any information about their correlation. X, Y may or may not be correlated.

76. The pdf of joint X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 1 & 0 \le x, y \le 1 \\ 0 & o/w \end{cases}$$

(a) We will use Jacobian transformation to find the probability density function at U, V space. The the non-zero probability region from original $X - Y$ space to $U - V$ space for each problem is indicated by the shaded region S_1, S_2, S_3.



i. We have $x = \frac{uv}{v+1}, y = \frac{u}{v+1}$, where u, v is at region S_1. The Jacobian determinant is

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ y^{-1} & -xy^{-2} \end{vmatrix} = -xy^{-2} - y^{-1} = \frac{-(v+1)^2}{u}$$

The probability density function at X, Y space is scaled down by the absolute value of the Jacobian determinant at the new space U, V.

$$f_{U,V}(u,v) = \begin{cases} \frac{u}{(v+1)^2} & u,v \in S_1 \\ 0 & o/w \end{cases}$$

ii. We have $x = u, y = \frac{u}{v}$. The Jacobian determinant is,

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ y^{-1} & -xy^{-2} \end{vmatrix} = -xy^{-2} = -\frac{v^2}{u}$$

The pdf becomes,

$$f_{U,V}(u,v) = \begin{cases} \frac{u}{v^2} & u,v \in S_2 \\ 0 & o/w \end{cases}$$

iii. We have $x = uv, y = u - uv$. The Jacobian determinant is,

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{y}{(x+y)^2} & \frac{-x}{(x+y)^2} \end{vmatrix} = \frac{-1}{x+y} = \frac{1}{u}$$

The pdf becomes,

$$f_{U,V}(u,v) = \begin{cases} u & u,v \in S_3 \\ 0 & o/w \end{cases}$$

(b) The marginal pdf of $U = X + Y$ from (i) can be found by integrating along the v-axis.

$$f_U(u) = \begin{cases} \int_0^\infty \frac{u}{(v+1)^2} dv = u & 0 \le u < 1 \\ \int_{u-1}^{\frac{1}{u-1}} \frac{u}{(v+1)^2} dv = 2 - u & 1 \le u \le 2 \\ 0 & o/w \end{cases}$$

(c) In the same way, the marginal pdf of U from (iii) is

$$f_U(u) = \begin{cases} \int_0^1 u dv = u & 0 \le u < 1 \\ \int_{1-1/u}^{1/u} u dv = 2 - u & 1 \le u \le 2 \\ 0 & o/w \end{cases}$$

We found the same answer as (b) regardless of the transformation of V.

77. Since X, Y, Z are iid exponential r.v. of $\lambda = 1$, the joint pdf is

$$f_{X,Y,Z}(x,y,z) = \begin{cases} e^{-(x+y+z)} & 0 \le x,y,z \\ 0 & o/w \end{cases}$$

And $X = \frac{U+V-W}{2}, Y = \frac{W+U-V}{2}, Z = \frac{V+W-U}{2}$. Using Jacobian transform of three r.v.s, we have the scalar

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2$$

Therefore, the new pdf at U, V, W space is

$$f_{U,V,W}(u,v,w) = \begin{cases} \frac{1}{2} e^{-(u+v+w)/2} & |u-v| \le w \le u+v, 0 \le u,v,w \\ 0 & o/w \end{cases}$$

78. We know the best mean square estimator for X^3 given Y is $g(Y) = \mathsf{E}[X^3|Y]$. To find it, we first find the conditional pdf of X given $Y = y$, which is $f_{X|Y}(x|y) = \frac{e^{-(x/y)}}{y}$ for x and y both positive. Then, we have $\mathsf{E}[X^3|Y] = \int_0^\infty x^3 \frac{e^{-(x/y)}}{y} dx = 6y^3$ for y is positive.

79. (a)

$$
\begin{aligned}
f(a,b,c) = \mathsf{E}[(X - a - bY - cZ)^2] = \\
&\mathsf{E}[X^2] - b\mathsf{E}[XY] - c\mathsf{E}[XZ] - a\mathsf{E}[X] + b^2\mathsf{E}[Y^2] - b\mathsf{E}[XY] + cb\mathsf{E}[YZ] + ab\mathsf{E}[Y] \\
&- c\mathsf{E}[XZ] + cb\mathsf{E}[ZY] + c^2\mathsf{E}[Z^2] + ba\mathsf{E}[Z] - a\mathsf{E}[X] + bc\mathsf{E}[Y] + ac\mathsf{E}[Z] + a^2
\end{aligned}
$$

To find the minimum, we take partial differential on all three variables and make them equal zero. Then we have three equations with three unknown.

$$
\begin{aligned}
\frac{\partial f}{\partial a} &= 0 \quad \mathsf{E}[X] = a + b\mathsf{E}[Y] + c\mathsf{E}[Z] \\
\frac{\partial f}{\partial b} &= 0 \quad \mathsf{E}[XZ] = a\mathsf{E}[Z] + b\mathsf{E}[YZ] + c\mathsf{E}[Z^2] \\
\frac{\partial f}{\partial c} &= 0 \quad \mathsf{E}[XY] = a\mathsf{E}[Y] + b\mathsf{E}[Y^2] + c\mathsf{E}[YZ]
\end{aligned}
$$

Solving for a^*, b^*, c^*, we have

$$
\begin{aligned}
a^* &= \mathsf{E}[X] - b^*\mathsf{E}[Y] - c^*\mathsf{E}[Z] \\
b^* &= \frac{\mathsf{cov}(Y,Z)\mathsf{cov}(X,Z) - \mathsf{cov}(X,Y)\mathsf{var}(Z)}{(\mathsf{cov}(Y,Z))^2 - \mathsf{var}Y\mathsf{var}Z} \\
c^* &= \frac{\mathsf{cov}(Y,Z)\mathsf{cov}(X,Y) - \mathsf{cov}(X,Z)\mathsf{var}(Y)}{(\mathsf{cov}(Y,Z))^2 - \mathsf{var}Y\mathsf{var}Z}
\end{aligned}
$$

Alternative solution: The orthogonal principle says, if the mean square error for some a^*, b^*, c^* from r.v. X to the linear r.v. vector space $a + bY + cZ$ spanned by r.v. $1, Y, Z$ is minimized, the error vector from X to $a^* + b^*Y + c^*Z$ must be orthogonal to the space, thus all basis of the space. Therefore, $X - a^* - b^*Y - c^*Z$ is orthogonal to $1, Y, Z$. In this sense, the following equations are hold.

$$
\begin{aligned}
\mathsf{E}[X - a^* - b^*Y - c^*Z] &= 0 && (1) \\
\mathsf{E}[(X - a^* - b^*Y - c^*Z)Y] &= 0 && (2) \\
\mathsf{E}[(X - a^* - b^*Y - c^*Z)Z] &= 0 && (3)
\end{aligned}
$$

Immediately, using (1), we get

$$a^* = E[X] - b^*\mathsf{E}[Y] - c^*\mathsf{E}[Z] \tag{4}$$

Using (1) and $\mathsf{cov}(X,Y) = \mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y]$, (2) and (3) becomes

$$
\begin{aligned}
\mathsf{cov}(X - a^* - b^*Y - c^*Z, Y) &= \mathsf{cov}(X - b * Y - c^*Z, Y) = 0 && (5) \\
\mathsf{cov}(X - a^* - b^*Y - c^*Z, Z) &= \mathsf{cov}(X - b * Y - c^*Z, Z) = 0 && (6)
\end{aligned}
$$

Further, (5) (6) can be simplified by the linearity of covariance.

$$
\begin{aligned}
\mathsf{cov}(X,Y) - b^*\mathsf{var}(Y) - c^*\mathsf{cov}(Z,Y) &= 0 && (7) \\
\mathsf{cov}(X,Z) - b^*\mathsf{cov}(Y,Z) - c^*\mathsf{var}(Z) &= 0 && (8)
\end{aligned}
$$

Solving for b^*, c^*, we have

$$b^* = \frac{\mathsf{cov}(Y,Z)\mathsf{cov}(X,Z) - \mathsf{cov}(X,Y)\mathsf{var}(Z)}{(\mathsf{cov}(Y,Z))^2 - \mathsf{var}Y\,\mathsf{var}Z} \tag{9}$$

$$c^* = \frac{\mathsf{cov}(Y,Z)\mathsf{cov}(X,Y) - \mathsf{cov}(X,Z)\mathsf{var}(Y)}{(\mathsf{cov}(Y,Z))^2 - \mathsf{var}Y\,\mathsf{var}Z} \tag{10}$$

The value a^*, b^*, c^* from (4) (9) (10) minimizes $\mathsf{E}[(X - (a + bY + cZ)]$.

(b) Let $Z = Y^2$ in the solution of (a) to get the best quadratic estimator for (b).