15. N is a discrete random variable.

$$\begin{aligned}
E[N] &= \sum_{n=1}^{\infty} n \Pr(N=n) \\
&= \Pr(N=1) + \Pr(N=2) + \Pr(N=2) + \cdots + \underbrace{\Pr(N=n) + \cdots + \Pr(N=n)}_{n} + \cdots \\
&= \sum_{n=1}^{\infty} \Pr(N=n) + \sum_{n=2}^{\infty} \Pr(N=n) + \cdots + \sum_{n=i}^{\infty} \Pr(N=n) + \cdots \\
&= \Pr(N \geq 1) + \Pr(N \geq 2) + \cdots + \Pr(N \geq i) + \cdots \\
&= \sum_{n=1}^{\infty} \Pr(N \geq n)
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{\infty} n \Pr(N > n) &= \Pr(N=2) + \Pr(N=3) + \Pr(N=4) + \cdots + \Pr(N=n) + \cdots \\
&\quad + 2\Pr(N=3) + 2\Pr(N=4) + \cdots + 2\Pr(N=n) + \cdots \\
&\quad + 3\Pr(N=4) + \cdots + 3\Pr(N=n) + \cdots \\
&\quad \cdots \\
&\quad + (n-2)\Pr(N=n-1) + (n-2)\Pr(N=n) + \cdots \\
&\quad + (n-1)\Pr(N=n) + \cdots \\
&= \Pr(N=2) + 3\Pr(N=3) + \cdots + \frac{n(n-1)}{2}\Pr(N=n) + \cdots \\
&= \sum_{n=2}^{\infty} \frac{1}{2}(n^2 \Pr(N=n) - n\Pr(N=n)) + \Pr(N=1) - \Pr(N=1) \\
&= \frac{1}{2}(\sum_{n=1}^{\infty} n^2 \Pr(N=n) - \sum_{n=1}^{\infty} n \Pr(N=n))
\end{aligned}$$

Using LOTUS, we have $E[N^2] = \sum_{n=1}^{\infty} n^2 \Pr(N=n)$, then

$$= \frac{1}{2}(E[N^2] - E[N])$$

16. (a) Let p_Y be the pmf of Y. Since g is a one-to-one function, for any x there exists a unique $y = g(x)$ such that $p_X(x) = p_Y(y)$. This is true because the same probability measure is defined on the function projection of X. Let the random variable $Y = g(X)$ be the projection space of X via function g. We have,

$$\begin{aligned}
E[g(X)] = E[Y] &= \sum_{y} y p_Y(y) \\
&= \sum_{g(x)} g(x) p_Y(g(x))
\end{aligned}$$

Since $p_Y(g(x)) = p_X(x)$ for all x, we have

$$= \sum_x g(x) p_X(x)$$

(b) If g is a many-to-one function, for all $x_1, x_2, \cdots, x_n$ such that $y = g(x_1) = g(x_2) = \cdots = g(x_n)$, we have $p_Y(y) = \sum_{x_i} p_X(x_i)$. The probability mass for these y's that have many to one mappings is the sum of the mass of all inverse imagines. Therefore,

$$\begin{aligned} E[g(X)] = E[Y] &= \sum_y y p_Y(y) \\ &= \sum_y \sum_{x_i, y=g(x_i)} g(x_i) p_X(x_i) \\ &= \sum_x g(x) p_X(x) \end{aligned}$$

17. Using LOTUS (the result from 16), we get

$$E[Y] = E[\frac{X-\mu}{\sigma}] = \frac{E[X]-\mu}{\sigma} = 0$$

We know that $\mathsf{var}(X) = E[X^2] - (E[X])^2$. So $E[X^2] = \sigma^2 + \mu^2$. By using this, we have

$$E[Y^2] = E[\frac{1}{\sigma^2}(X^2 - 2X\mu + \mu^2)] = \frac{1}{\sigma^2}(E[X^2] - 2E[X]\mu + \mu^2) = 1$$

18. (a) At each round, let R be the event that red shows. The sample space for a game of maximum three tries is shown in the first row of the following table. For each outcome, we compute net earning and the probability.

Ω	R	R^c, R, R	R^c, R^c, R^c	R^c, R^c, R	R^c, R, R^c
X	\$1	\$1	-\$3	-\$1	-\$1
$\Pr[X]$	$\frac{18}{38}$	$\frac{20}{38}\frac{18}{38}\frac{18}{38}$	$\frac{20}{38}\frac{20}{38}\frac{20}{38}$	$\frac{20}{38}\frac{20}{38}\frac{18}{38}$	$\frac{20}{38}\frac{18}{38}\frac{20}{38}$

For the winning probability, we sum the outcomes that $X > 0$. $\Pr(X > 0) = \frac{18}{38} + \frac{20}{38}\frac{18}{38}\frac{18}{38} = 0.5918$.

(b) You have more than 50% chance to win the game. However the answer from (c) shows that you lose 11 cents on average when you play the game.

(c) $E[X] = \frac{18}{38} + \frac{20}{38}\frac{18}{38}\frac{18}{38} - 3\frac{20}{38}\frac{20}{38}\frac{20}{38} - \frac{20}{38}\frac{20}{38}\frac{18}{38} - \frac{20}{38}\frac{18}{38}\frac{20}{38} = -\frac{39}{361} = -0.1080$

19. (a) Show that $0 \le p_X(k) \le 1$. We noticed that $p_X(k)$ is a monotonously increasing function with k and $p_X(k)$ is always greater than 0. When $k = n$, $p_X(n) = \frac{2}{n+1}$. As $n \ge 1$, the largest value $p_X(n) \le 1$. Then we need to show that the probability mass is summed up to 1. $\sum_{k=1}^n p_X(k) = \frac{2}{n(n+1)}\frac{n(n+1)}{2} = 1$. Therefore, p_X is a valid pmf.

(b) The range of Y is all the inverse of integers from $[1, n]$. The pmf of Y is

$$p_Y(k) = \Pr(Y = k) = \Pr(1/X = k) = \Pr(X = 1/k) = \frac{2}{kn(n+1)}$$

with $k = 1/i$ where $1 \le i \le n$ is an integer.

(c) $E[Y] = \sum_k k p_Y(k) = \frac{2}{n+1}$. Or by LOTUS, $E[Y] = \sum_k \frac{1}{k} p_X(k)$ leads to the same answer.

20. (a) The Taylor series expansion of function e^λ at $x=0$ (or Maclaurin series) is $e^\lambda = 1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots+\frac{\lambda^k}{k!}+\cdots$. Therefore, the sum of the pmf $p_X(k)=\sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda}=1$.

(b)

$$E[X] = \sum_{k=0}^{\infty} e^{-\lambda} k \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} e^{-\lambda}\lambda \frac{\lambda^{k-1}}{(k-1)!}$$

By using the fact from (a) that $\sum_{k=1}^{\infty} e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!} = 1$

$$E[X] = \lambda$$

(c) Again by LOTUS,

$$\begin{aligned} E[X(X-1)] &= \sum_{k=0}^{\infty} e^{-\lambda} k(k-1)\frac{\lambda^k}{k!} \\ &= \sum_{k=2}^{\infty} e^{-\lambda}\lambda^2 \frac{\lambda^{k-2}}{(k-2)!} \\ &= \lambda^2 \end{aligned}$$

(d) $\mathsf{var}(X) = E[X^2] - (E[X])^2 = E[X^2 - X] + E[X] - (E[X])^2 = \lambda$

(e) By LOTUS,

$$\begin{aligned} E[z^X] &= \sum_{k=0}^{\infty} z^k e^{-\lambda}\frac{\lambda^k}{k!} \\ &= \sum_{k=0}^{\infty} e^{-\lambda}\frac{(\lambda z)^k}{k!} \\ &= e^{-\lambda}e^{z\lambda} \\ &= e^{\lambda(z-1)} \end{aligned}$$