21. (a) Let $X \sim B(n,p)$ be a binomial r.v. for the number of judges who vote for guilty, where n is the number of judges and p is the probability that a judge votes for guilty.When a defendant is guilty, $p = 0.7$.

    i. $X \sim B(9, 0.7)$. If more than 4 judges vote for guilty, the defendant is declared guilty. $\Pr(X \geq 5) = \sum_{i=5}^{9} \binom{9}{i} 0.7^i 0.3^{9-i} = 0.9012$

    ii. $X \sim B(8, 0.7)$. If more than 5 judges (or 4 judges depending on the definition of majority) vote for guilty, the defendant is declared guilty. $\Pr(X \geq 5) = \sum_{i=5}^{8} \binom{8}{i} 0.7^i 0.3^{8-i} = 0.8059$. Or $\Pr(X \geq 4) = \sum_{i=4}^{8} \binom{8}{i} 0.7^i 0.3^{8-i} = 0.9420$

    iii. $X \sim B(7, 0.7)$. If more than 4 judges vote for guilty, the defendant is declared guilty. $\Pr(X \geq 4) = \sum_{i=4}^{7} \binom{7}{i} 0.7^i 0.3^{7-i} = 0.8740$

    (b) The defense attorney needs to find a strategy that minimizes the probability that the client is declared guilty. For each strategy, the attorney will compute the probability by Pr(guilty defendant declared guilty)Pr(guilty client)+Pr(innocent defendant is declared guilty)Pr(innocent client), where Pr(guilty client) is 0.6 and Pr(innocent client) is 0.4.

    From (a) we have computed the probability for a guilty defendant to declare guilty, in a similar way we can find the probability for an innocent defendant to be declared guilty with $p = 0.3$. The answer for an innocent client to be declared guilty is 0.0988 for 9 judges, 0.0580(0.1941) for 8 judges and 0.1260 for 7 judges.

    If the attorney does not exercise peremptory challenge right, the probability that the client is declared guilty is $0.9012 \cdot 0.6 + 0.0988 \cdot 0.4 = 0.5802$. If the attorney removes one judge, the probability is $0.8059 \cdot 0.6 + 0.0580 \cdot 0.4 = 0.5067$ or 0.6428 if majority is 4 out of 8. If the attorney removes two judges, the probability is 0.5748. To minimize the probability, the attorney should choose to remove one judge if majority is 5 out of 8. Otherwise, the attorney should remove two judges.

22. Intuitively, the probability of rejecting a lot should not depend on the size of lot since the failure rate for each item is independent. If a purchaser examines 4 components, the probability that all 4 are good is $0.9^4 = 0.6561$. So the rejection rate is $1 - 0.6561 = 0.3439$.

    Then we show that our intuition is true. Let n be the number of components in a lot. For a lot with k defective components for $k \leq n-4$, the probability that the purchaser accepts the lot equals Pr(k defective components in n) Pr(the purchaser chooses 4 good ones). It is

$$\binom{n}{k} 0.1^k 0.9^{n-k} \frac{\binom{n-k}{4}}{\binom{n}{4}} = \binom{n-4}{k} 0.1^k 0.9^{n-k}$$

    The right hand side means that all defect components fall into the set that the purchaser does not select. It is obvious that $k \leq n-4$ otherwise the purchaser always gets defect components. To sum them up for all k's, $\sum_{k=0}^{n-1} \binom{n-4}{k} 0.1^k 0.9^{n-4-k+4} = 0.9^4$. The result is thus proved.

23. Let X be the r.v. standing for the number of passengers left behind. Note that the maximum value of X is 3. The probability of k passengers left behind for $k > 0$ is the probability that $k+4$ passengers show up. When $k = 0$, the probability is that for $k \leq 4$ passengers showing up. As $k > 0$, $\Pr(X = k) = \binom{7}{k+4} 0.5^{k+4} 0.5^{3-k}$. The expectation of the number of passengers left behind is $E(X) = \sum_{k=1}^{3} k \Pr(X = k) = 0.1641 + 0.0547 \cdot 2 + 0.0078 \cdot 3 = 0.2969$.

24. (a) Since you have to toss the coin at least twice to get two Heads and you may toss an arbitrary number of times to get two Heads, N is an integer from 2 to positive infinity.

    (b) $\Pr(N = n) = \binom{n-1}{1} p(1-p)^{n-2} p = (n-1)p^2(1-p)^{n-2}$ for $n = 2, 3, \ldots$.

(c) Using ML estimation, we need to find the probability p that maximizes the probability mass at $N = n$. To find the critical points in the continuous function $f(p) = (n-1)p^2(1-p)^{n-2}$, we do the first derivative test. $\frac{df(p)}{dp} = 0$ and solve it we get $p = 0, \frac{2}{n}, 1$. Then we need to show that the function is increasing in the region $p \in [0, \frac{2}{n}]$ and decreasing in the other. $f(0) = f(1) = 0$ and $f(2/n) > 0$ so $p = \frac{2}{n}$ is the maximum. Or we can do second derivate test to show that $\frac{d^2f}{dp^2} < 0$ at $p = \frac{2}{n}$.

25. (a) The number of Heads that will occur on the next 1000 trials follows a binomial distribution with $n = 1000, p = 0.11$. We guess the mean value $np = 110$. The probability is $\binom{1000}{110}0.11^{110}0.89^{890} = 0.0403$. Or we could guess the most likely value. To find the maximum value we use ratio test method.

$$\frac{\binom{1000}{k}0.11^k 0.89^{1000-k}}{\binom{1000}{k+1}0.11^{k+1}0.89^{1000-k-1}} \geq 1$$

Solving for it we also get $k = 110$.

(b) The number of tosses made to observe the next Head (includes the toss to get the Head) follows a geometric distribution with $p = 0.11$. If the king wants the exact number of tosses, we should guess 1 where the maximum probability mass occurs. The probability is 0.11. Or we could guess mean value $1/p \approx 9$ where probability is $0.89^8 0.11 = 0.0433$.

(c) The number of tosses made to observe the 105th Head follows a negative binomial distribution with $r = 105, p = 0.11$. We could guess the mean value $r/p \approx 955$ where the probability is $\binom{954}{104}0.11^{105}0.89^{850} = 0.0045$. Or we could find the most likely value. Again using the ratio method.

$$\frac{\binom{k-1}{104}0.11^{105}0.89^{k-105}}{\binom{k}{104}0.11^{105}0.89^{k+1-105}} \geq 1$$

Solving for it we get $k = 946$. The probability mass is 0.0046.

26. For Binomial r.v., $\mathsf{E}[X] = np = 6, \mathsf{var}(X) = np(1-p) = 2.4$. Solve it we have $p = 0.6, n = 10$.

(a) $\Pr(X = 5) = \binom{10}{5}0.6^5 0.4^5 = 0.2007$

(b) $\Pr(2X \geq 5) = \Pr(X \geq 2.5) = \sum_{k=3}^{10} \binom{10}{k}0.6^k 0.4^{10-k} = 0.9877$

(c) $\mathsf{var}(3X - 2) = 9\mathsf{var}(X) = 21.6$

(d) Using the equation $\mathsf{E}[X^3] = np\mathsf{E}[(Y+1)^2]$ where Y is $\mathsf{Binom}(n-1, p)$. Hence, $\mathsf{E}[X^3] = 6\mathsf{E}[Y^2 + 2Y + 1] = 6((n-1)(n-2)p^2 + (n-1)p + 2(n-1)p + 1) = 6(72(.36) + 3(5.4) + 1) = 258.72$. In general, we could use z-transform method for arbitrary distributions. For Binomial r.v, the z-transform is $\mathsf{E}[z^X] = \sum_{k=0}^{n} \binom{n}{k}(zp)^k(1-p)^k = (1-p+pz)^n$.

$$\frac{d^3}{dz^3}\mathsf{E}[z^X]|_{z=1} = \mathsf{E}[X(X-1)(X-2)] = \mathsf{E}[X^3] - 3\mathsf{E}[X^2] + 2\mathsf{E}[X]$$

We also have

$$\frac{d^3}{dz^3}(1-p+pz)^n|_{z=1} = n(n-1)(n-2)p^3 = 155.52$$

We know that $\mathsf{E}[X^2] = \mathsf{var}(X) + \mathsf{E}[X]^2 = 38.4$. So $\mathsf{E}[X^3] = 155.52 - 2\cdot 6 + 3\cdot 38.4 = 258.72$.

27. Since $nR_n \sim \mathsf{Binom}(n, P(A))$, $\mathsf{var}(R_n) = P(A)(1-P(A))/n = 0.0099/n$. From Chebyshev's inequality, $P(|R_n - 0.01| \geq 0.001) \leq 0.0099/0.000001n$. Since we are 99% confident that $|R_n - 0.01|$ falls in the range of 0.001, we want the RHS to be less than 0.01, that is $9900 < 0.01n$, which leads to $n > 990,000$.