33. (a) Let A be the event that two consecutive coin flips have different faces. Let H be the event that the last flip landed on heads. Then, $P(A) = 2p(1-p), P(HA) = p(1-p)$. So the probability to get a heads in this procedure is $P(H|A) = P(HA)/P(A) = 1/2$.
    (b) No. Simplified procedure is generating the probability that the last flip is a heads given previous flip is a tails, which is the same as p.

34. The probability that E happens before F is the probability that E happens given E or F happens. $P(E \cup F) = P(E) + P(F), P(E|E \cup F) = P(E)/P(E \cup F) = P(E)/(P(E) + P(F))$.

35. We have,
$$P(\bigcup_{i=1}^{n} E_i) = 1 - P(\bigcap_{i=1}^{n} E_i^c) = 1 - \prod_{i=1}^{n} P(E_i^c) = 1 - \prod_{i=1}^{n}(1 - P(E_i))$$
where in the first equality we used De Morgan's Law, and the second equality follows from the independence of E_i's. The third one follows because $P(E_i^c) = 1 - P(E_i)$.

36. We are given that the probability that a coin lands on Heads is $p = 1/2$. $P(A) = 1/2, P(B) = 1/2, P(C) = p^2 + (1-p)^2 = 1/2$. To show n events are independent, it is necessary to show that $P(\bigcap_{i=1}^{m} E_i) = \prod_{i=1}^{m} P(E_i)$, for all $m \leq n$. To check pairwise independence, $P(AB) = p^2 = 1/4 = P(A)P(B)$, $P(AC) = p^2 = 1/4 = P(A)P(C), P(BC) = p^2 = P(A)P(C)$. So they are pairwise independent. However, $P(ABC) = p^2 = 1/4 \neq P(A)P(B)P(C) = 1/8$. So they are not independent.

37. (a) Given 1, 2, 3 or 4 is drawn, the probability that 3, 4, or 5 is drawn equals to the probability that 3, 4 is draw amongst 1, 2, 3, 4. So $P(E_1|F) = 1/2$. Similarly, $P(E_2|F) = 1/2$. Further, $P(E_1E_2|F)$ is the probability that 4 is drawn amongst 1, 2, 3, 4, which is equal to 1/4. Then, $P(E_1E_2|F) = P(E_1|F)P(E_2|F)$. They are conditionally independent.
    (b) Drawing 3, 4, or 5 from 1-6, we have the probability $P(E_1) = 1/2$. Similarly, $P(E_2) = 1/2$, and $P(E_1E_2) = 1/6$. But $P(E_1E_2) \neq P(E_1)P(E_2)$. Hence, they are not independent.
    (c) Five balls, $E_1 = \{1, 2\}, E_2 = \{2, 3\}, F = \{1, 2, 3, 4\}$. We could show that E_1, E_2 are conditionally independent on given F but not unconditionally independent.

38. (a) For any given batch size N, the number of defective components k in line A follows Binomial distribution $\mathsf{Binom}(p_0, N)$ and those in line B follows $\mathsf{Binom}(p_1, N)$. When $N = 100$, the probability that $k = 7$ in line A is 0.1060 and the probability that $k = 7$ in line B is 0.0889. The likelihood matrix is

	$k = 7$
A	0.1060
B	0.0889

    (b) The ML decision is that the batch is produced by line A.
    (c) The ML decision rule becomes,
$$\frac{\binom{N}{k} p_0^k (1-p_0)^{N-k}}{\binom{N}{k} p_1^k (1-p_1)^{N-k}} = 0.5^k \frac{19}{18}^{N-k} \underset{B}{\overset{A}{\gtrless}} 1$$
Taking the logarithm of both sides (since logarithm is a monotonically increasing function):
$$-k \log 2 + (N-k) \log \frac{19}{18} \underset{B}{\overset{A}{\gtrless}} 0$$
$$k \underset{A}{\overset{B}{\gtrless}} N \frac{\log 19 - \log 18}{\log 19 - \log 9} = 0.0724N$$

39. (a) Let X be the number of Heads in 5 flips. Again $X \sim \mathsf{Binom}(5, 0.5)$ for a fair coin and $X \sim \mathsf{Binom}(5, 0.7)$ for a biased coin. The ML matrix and decision rule is (given by the underlined entries):

	$X=0$	$X=1$	$X=2$	$X=3$	$X=4$	$X=5$
H_0	$\underline{0.0312}$	$\underline{0.1562}$	$\underline{0.3125}$	$\underline{0.3125}$	0.1562	0.0312
H_1	0.0024	0.0283	0.1323	0.3087	$\underline{0.3601}$	$\underline{0.1681}$

Hence the ML decision rule is to decide that the coin is biased when 4 or more Heads appear. The error probability is $P(e) = P(X \geq 4|H_0)P(H_0) + P(X \leq 3|H_1)P(H_1) = 0.1874(0.8) + (0.4717)(0.2) = 0.2443$.

(b) We know that $\pi_0 = P(H_0) = 0.8, \pi_1 = P(H_1) = 0.2$. So the joint probability matrix and MAP decision rule is

	$X=0$	$X=1$	$X=2$	$X=3$	$X=4$	$X=5$
H_0	$\underline{0.0250}$	$\underline{0.1250}$	$\underline{0.2500}$	$\underline{0.2500}$	$\underline{0.1250}$	0.0250
H_1	0.0005	0.0057	0.0265	0.0617	0.0720	$\underline{0.0336}$

The MAP rule dictates that we should rule that the coin is biased only when 5 heads appear. the error probability is obtained by adding up all the entries in the table above which are not underlined, which leads to $P(e) = 0.1914$.

(c) For the Bayes' minimum average cost decision rule, the threshold is:

$$\tau = \frac{(C_{10} - C_{00})\pi_0}{(C_{01} - C_{11})\pi_1} = \frac{8}{3} = 2.6667$$

If $\Lambda(X) > \tau$, the decision is H_1. Otherwise, the decision is H_0. Finding the ratio in the likelihood matrix and make the decision,

	$X=0$	$X=1$	$X=2$	$X=3$	$X=4$	$X=5$
$\Lambda(X)$	0.0778	0.1814	0.4234	0.9878	2.3050	5.3782
H_0	$\underline{0.0312}$	$\underline{0.1562}$	$\underline{0.3125}$	$\underline{0.3125}$	$\underline{0.1562}$	0.0312
H_1	0.0024	0.0283	0.1323	0.3087	0.3601	$\underline{0.1681}$

Hence, the Bayes minimum average cost decision rule dictates that the coin be declared to be biased only if 5 heads appear.