40. (a) According to the law of total probability, we have $\int_{-1}^{1} c(1-x^2)dx = c(x - \frac{x^3}{3})|_{-1}^{1} = 1$. Solving for c, we get $c = \frac{3}{4}$.

(b)

$$F(x) = \begin{cases} -\frac{x^3}{4} + \frac{3}{4}x + \frac{1}{2} & -1 < x < 1 \\ 1 & x \geq 1 \\ 0 & otherwise \end{cases}$$

41. (a) $P(X > 20) = \int_{20}^{\infty} \frac{10}{x^2} dx = -10x^{-1}|_{20}^{\infty} = \frac{1}{2}$

(b)

$$F(x) = \begin{cases} 1 - \frac{10}{x} & x > 10 \\ 0 & otherwise \end{cases}$$

(c) The probability of one device that operates at least 15 hours is $p = P(X \geq 15) = -10x^{-1}|_{15}^{\infty} = \frac{2}{3}$. Assuming that all six devices operate independently, the number of devices that fail, Y, follows Binomial distribution $B(p, 6)$. Then $P(Y \geq 3) = \sum_{i=3}^{6} \binom{6}{i} p^i (1-p)^{6-i} = 0.8999$.

42. Solution 1: If two cards share the same random variable X, the threshold is 1000. Define r.v.'s Y_1 and Y_2 as the payoffs marked on the card you first picked and that on the card you can switch to, respectively. The optimal strategy is to switch when the expected outcome conditional on $Y_1 = y$ is greater than y. There are two cases:

- $y > 1000$, no need to switch.
- $0 \leq y \leq 1000$, note that

$$P(Y_2 = 2y | Y_1 = y) = \frac{2}{3} . P(Y_2 = \frac{y}{2} | Y_1 = y) = \frac{1}{3}$$

Note that the conditional probability on y stays the same regardless of the type of r.v. X. Thus, the expected payoff conditional on $Y_1 = y$ is

$$2y \cdot \frac{2}{3} + \frac{y}{2} \cdot \frac{1}{3} = \frac{3}{2} y$$

which is greater than y. Therefore, the optimal strategy is always to switch to the other card.

Solution 2 under a different interpretation: If two cards use different random variables generated independently, the threshold is 833. Again no need to switch if $y > 1000$. Let D be the event that the first card drawn follows r.v. X. If $y \leq 1000$, $P(D) = \frac{2}{3}$, $P(D^c) = \frac{1}{3}$. Then, the expected payoff of switching a card is

$$E[2X]P(D) + E[X]P(D^c) = 1000\frac{2}{3} + 500\frac{1}{3} = 833$$

Therefore, one should switch card if $y < 833$.

43. (a) The pmf for N is

$$p_N(n) = p_\Delta (1 - p_\Delta)^{n-1}, \quad \forall n \in \mathcal{N}$$

The cdf is

$$F_N(n) = 1 - (1 - p_\Delta)^n, \quad \forall n \in \mathcal{N}$$

(b) Suppose $p_\Delta = \lambda\Delta$ for some proportionality constant λ. We have

$$P(T \le n\Delta) = P(N \le n) = 1 - (1 - \lambda\Delta)^n, \quad \forall n \in \mathcal{N}$$

As $\Delta \to 0$, $t = n\Delta$ is approximately continuous and we have

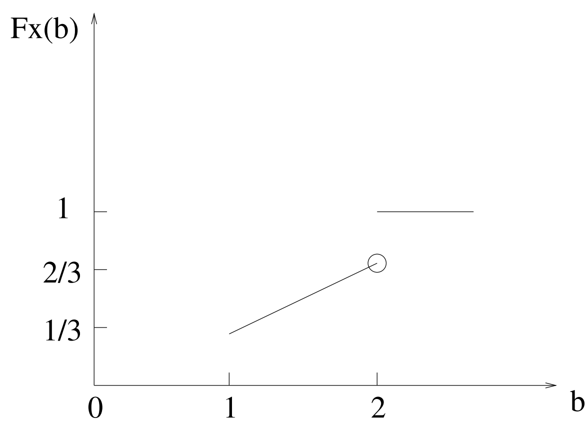
$$F_T(t) \approx 1 - e^{-\lambda n\Delta} = 1 - e^{-\lambda t}, \quad t \ge 0$$

which is the cdf of an exponential random variable. Thus, T is approximately exponentially distributed with parameter λ.

44. Define r.v. X as the number of damaged packages. The distribution of X can be approximated by a Poisson random variable with parameter $\lambda = 10^6 \times 0.01 = 10^4$. Thus, we have

$$P(X > 10200) = 1 - \sum_{i=0}^{10200} \frac{\lambda^i e^{-\lambda}}{i!} = 0.0227$$

45. (a) It is a mixed random variable.



(b) $P(|X-1| < 1) = P(0 < X < 2) = F(2^-) - F(0) = \frac{2}{3}$

(c)

$$\begin{aligned} P(|X-1| < 1 | 1 < X \le 2) &= \frac{P(1 < X < 2)}{P(1 < X \le 2)} \\ &= \frac{F(2^-) - F(1)}{F(2^+) - F(1)} \\ &= \frac{2/3 - 1/3}{1 - 1/3} = 1/2 \end{aligned}$$