University of Illinois, Urbana–Champaign Fall 2007
ECE 313: Solutions to Homework 8

46. Let r.v. $U \sim Exp(3)$ and $V \sim Exp(6)$, then $f_X = \frac{2}{3} f_U + \frac{1}{3} f_V$.

(a) Checking for non-negative property, $f_X(x) \geq 0$ for all x. The total probability is $\int_0^\infty f_X(b)db = 2/3 + 1/3 = 1$. Therefore, f_X is a valid pdf.

(b)

$$\begin{aligned}
\mathsf{E}[X] &= \frac{2}{3}\mathsf{E}[U] + \frac{1}{3}\mathsf{E}[V] = \frac{5}{18} \\
\mathsf{E}[X^2] &= \frac{2}{3}\mathsf{E}[U^2] + \frac{1}{3}\mathsf{E}[V^2] = \frac{1}{6} \\
\mathsf{var}(X) &= \mathsf{E}[X^2] - (\mathsf{E}[X])^2 = \frac{29}{324} = 0.0895
\end{aligned}$$

(c) $P(|X| < 2) = P(0 \leq X < 2) = \frac{2}{3}P(0 \leq U < 2) + \frac{1}{3}P(0 \leq V < 2) = 1 - \frac{2}{3}e^{-6} - \frac{1}{3}e^{-12} = 0.9983$

47. (a) The pdf for the continuous part of X is

$$f_X(b) = \begin{cases} \frac{1}{3} & 1 < b < 2 \\ 0 & otherwise \end{cases}$$

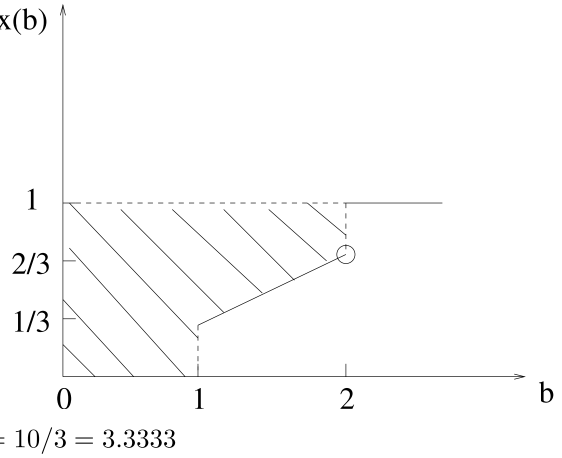
The pmf for the discrete part of X is

$$p_X(b) = \begin{cases} \frac{1}{3} & b = 2, b = 1 \\ 0 & otherwise \end{cases}$$

Note that the $\int_1^2 f_X(b)db + p_X(1) + p_X(2) = 1$. Then, we have

$$\begin{aligned}
\mathsf{E}[X] &= \int_1^2 \frac{b}{3}db + \frac{1}{3} + \frac{2}{3} = \frac{3}{2} = 1.5 \\
\mathsf{E}[X^2] &= \int_1^2 \frac{b^2}{3}db + \frac{4}{3} + \frac{1}{3} = \frac{22}{9} = 2.4444 \\
\mathsf{var}(X) &= \mathsf{E}[X^2] - (\mathsf{E}[X])^2 = \frac{7}{36} = 0.1944
\end{aligned}$$

$\mathsf{E}[X]$ can also be obtained by summing up the shaded area.



(b) $\mathsf{E}[3X^2 - 4] = 3\mathsf{E}[X^2] - 4 = 10/3 = 3.3333$

48. (a) According to the law of total probability, $\int_{-\infty}^{\infty} ce^{-2|x|}dx = \int_{-\infty}^{0} ce^{2x}dx + \int_{0}^{\infty} ce^{-2x}dx = c = 1$. Thus, $c = 1$.

    (b) By symmetry, $\mathsf{E}[X] = 0$.

$$\begin{aligned} \mathsf{var}(X) &= \mathsf{E}[X^2] = 2\int_0^\infty x^2 e^{-2x}dx = 1/2 \\ \mathsf{E}[Y] &= \mathsf{E}[3X-2] = -2 \\ \mathsf{var}(Y) &= 9\mathsf{var}(X) = 9/2 \end{aligned}$$

    (c)

$$\begin{aligned} P(Y>4) = P(3X-2>4) = P(X>2) &= \int_2^\infty e^{-2x}dx = \frac{e^{-4}}{2} \\ P(Y>0) = P(3X-2>0) = P(X>\frac{2}{3}) &= \frac{e^{-4/3}}{2} \\ P(Y>4|Y>0) = \frac{P(Y>4)}{P(Y>0)} &= e^{-8/3} = 0.0695 \end{aligned}$$

    (d) Since X is a continuous r.v except $X = 0$,

$$\begin{aligned} P(Y\geq 4|Y\geq 0) &= P(Y>4|Y>0) = e^{-8/3} \\ P(Y<4|Y\geq 0) &= 1 - P(Y\geq 4|Y\geq 0) = 1 - e^{-8/3} = 0.9305 \end{aligned}$$

49. (a)

$$\mathsf{E}[|X-a|] = \int_0^a (a-x)\frac{1}{A}dx + \int_a^A (x-a)\frac{1}{A}dx = \frac{\frac{A^2}{2} - aA + a^2}{A} = \frac{1}{A}((a-\frac{A}{2})^2 + \frac{A^2}{4})$$

    Letting the square part be zero to obtain minimum, we have $a = A/2$.

    (b)

$$\mathsf{E}[|X-a|] = \int_0^a (a-x)\lambda e^{-\lambda x}dx + \int_a^A (x-a)\lambda e^{-\lambda x}dx = a + \frac{2e^{-a\lambda}}{\lambda} + \frac{1}{\lambda}$$

    To find a that minimizes $\mathsf{E}[|X-a|]$, first we need to find the critical points by taking derivative of the RHS and let it equals zero. We got equation $1 - 2e^{-a\lambda} = 1$. Solving for a, we got $a = \frac{\ln 2}{\lambda}$. The second derivative $2\lambda e^{-a\lambda} > 0$ for $a = \frac{\ln 2}{\lambda}$. Thus it is the minimum.

50. (a) Let X be the life time of one light bulb. X follows exponential distribution with Poisson rate $\lambda = 10$ per hour. Since exponential distribution is memoryless, the probability that the bulb is working at 11pm only depends on the two hours after last check at 9pm, which equals to the probability p that a bulb continuously work during the past two hours, s.t. $p = P(X > 2) = \int_2^\infty 10e^{-10t}dt = e^{-20}$. The probability that all 5 bulbs are working at 11pm is $p^5 = e^{-100}$ since they are independent.

    (b) The probability that only two bulbs are still operating is $\binom{5}{2}p^2(1-p)^3 = 10e^{-40}(1 - e^{-20})^3 = 4.25 \times 10^{-17}$.

    (c) According to memoryless property, the probability that a bulb that was working at 11pm can last until midnight is $P(X > 1) = e^{-10}$. Thus the probability of two bulbs working is e^{-20}.

51. (a) A valid CDF should be monotonous nondecreasing, right-continuous and satisfying $F(-\infty) = 0, F(\infty) = 1$. Since F_1, F_2 both are valid CDFs that are nondecreasing and right-continuous, F_X must be nondecreasing and right-continuous. Further, $F_X(\infty) = 0.3F_1(\infty) + 0.7F_2(\infty) = 1, F_X(-\infty) = 0$. Since F_2 is not continuous, F_X is a CDF of a mixed random variable.

    (b) The pdf for the continuous part of X is

    $$f_X(t) = \begin{cases} 0.3 \times 2e^{-2t} & t \geq 0 \\ 0 & otherwise \end{cases}$$

    The pmf for the discrete part of X is

    $$p_X(t) = \begin{cases} 0.7 \times 0.4 & t = 1 \\ 0 & otherwise \end{cases}$$

    Then, we have

    $$\begin{aligned} \mathsf{E}[X] &= \int_0^\infty 0.6te^{-2t}dt + 0.28 = 0.43 \\ \mathsf{E}[X^2] &= \int_0^\infty 0.6t^2e^{-2t}dt + 0.28 = 0.43 \\ \mathsf{var}(X) &= 0.2451 \end{aligned}$$

    (c)

    $$\begin{aligned} P(0 \leq X < 2) &= P(0 < X \leq 2) = F_X(2) - F_X(0) \\ &= 0.3F_1(2) + 0.7F_2(2) - 0.3F_1(0) - 0.7F_2(0) \\ &= 0.3(1 - e^{-4}) + 0.7 = 0.9945 \\ P(0 < X \leq 2) &= 0.3F_1(2) + 0.7F_2(2) - 0.3F_1(0) - 0.7F_2(0^+) \\ &= 0.3(1 - e^{-4}) + 0.28 = 0.5745 \\ P(1 \leq X \leq 2) &= 0.3F_1(2) + 0.7F_2(2) - 0.3F_1(1) - 0.7F_2(1^-) \\ &= 0.3(e^{-2} - e^{-4}) + 0.28 = 0.3151 \\ P(1 < X \leq 2 | X \leq 2) &= \frac{P(1 < X \leq 2)}{P(X \leq 2)} = 0.0353 \end{aligned}$$

52. The quadratic equation has two distinct real roots if and only if $16\beta^2 - 16(\beta + 2) > 0$. Solving for the range of β satisfying that inequality, we have $(-\infty, -1)$ and $(2, \infty)$.

    (a) If $\beta \sim Unif(0, 5)$, the probability that this equation has two distinct real roots is $P(2 < \beta \leq 5) = 3/5 = 0.6$.

    (b) If $\beta \sim Exp(2)$, the probability becomes $P(\beta > 2) = e^{-4} = 0.0183$.

    (c) If $\beta \sim Poi(2)$, the probability becomes $P(\beta > 2) = 1 - P(\beta = 0) - P(\beta = 1) - P(\beta = 2) = 1 - e^{-2} - 2e^{-2} - 2e^{-2} = 0.3233$.