University of Illinois, Urbana–Champaign Fall 2007
ECE 313: Solutions to Homework 9

53. The life time of chip is denoted as normal r.v. X. The probability that X lasts for less than 1.8E6 hours is

$$P(X < 1.8E6) = P(\frac{X - 1.4E6}{3E5} < \frac{1.8E6 - 1.4E6}{3E5}) = \Phi(\frac{4}{3}) = 0.9088$$

Then, probability that at least 20 chips fail in 1.8E6 hours among 100 chips is

$$\sum_{i=20}^{100} (0.9088)^i (1 - 0.9088)^{100-i} \approx 1$$

54. The probability that the receiver detects -1 for a transmitted signal $X = 1$ is

$$P(Y < 0|X = 1) = P(a + Z < 0) = P(Z < -a) = P(\frac{Z}{\sigma_n} < \frac{-a}{\sigma_n}) = \Phi(\frac{-a}{\sigma_n}) = 1 - \Phi(\frac{a}{\sigma_n})$$

The probability that the receiver detects 1 for a transmitted signal $X = -1$ is

$$P(Y \geq 0|X = -1) = P(Z - a > 0) = P(Z > a) = 1 - \Phi(\frac{a}{\sigma_n})$$

Therefore, the total probability of error can be computed by Bayes' total probability law that

$$P(Y < 0|X = 1)P(x = 1) + P(Y \geq 0|X = -1)P(X = -1) = 1 - \Phi(\frac{a}{\sigma_n})$$

55. A r.v. Z following $\mathsf{Norm}(0, 1)$ distribution can be generated via r.v. X using $Z = \frac{X - \mu_1}{\sigma_1}$. Also, r.v. Y can be generated via Z using $Y = Z\sigma_2 + \mu_2$. Combining these two, we can get

$$Y = \frac{X - \mu_1}{\sigma_1}\sigma_2 + \mu_2$$

The procedure is to generate X first and use the above equation to get Y.

56. Let $N(t)$ denote the number of Poisson arrivals during time t.

    (a) $N(1)$ is the number of arrivals in the first hour. $N(\frac{1}{3})$ is the number of arrivals in the first 20 minutes. We need to find

    $$P(N(\frac{1}{3}) = 2|N(1) = 2) = \frac{P(N(\frac{1}{3}) = 2, P(N(1) - N(\frac{1}{3})) = 0)}{P(N(1) = 2)} = \frac{\frac{(\lambda/3)^2 e^{\frac{-\lambda}{3}}}{2!} e^{\frac{-2\lambda}{3}}}{\frac{\lambda^2 e^{-\lambda}}{2!}} = \frac{1}{9} = 0.1111$$

    Since the distribution of given number of Poisson arrivals in a certain interval is uniform, each arrival arrived during the first hour has $\frac{1}{3}$ chance to arrive in the first 20 minutes, also we get the probability $\frac{1}{9}$ for two independent arrivals.

    (b) Each arrival has $\frac{1}{3}$ chance to arrive within the first 20 minutes. The probability of number of arrivals in the first 20 minutes given two arrivals in a hour follows $\mathsf{Binom}(2, \frac{1}{3})$. Thus the probability that at least one arrived during the first 20 minutes is $2\frac{1}{3}\frac{2}{3} + \frac{1}{9} = \frac{5}{9} = 0.5556$ One can also directly compute the probability by

    $$P(N(\frac{1}{3}) \geq 1|N(1) = 2) = 1 - P(N(\frac{2}{3}) = 2|N(1) = 2) = 1 - \frac{P(N(\frac{1}{3}) = 0, P(N(\frac{2}{3})) = 2)}{P(N(1) = 2)}$$

    $$= 1 - \frac{e^{\frac{-\lambda}{3}} \frac{(2\lambda/3)^2 e^{\frac{-2\lambda}{3}}}{2!}}{\frac{\lambda^2 e^{-\lambda}}{2!}} = \frac{5}{9}$$

57. Assume that Dave will be injured if a car crosses the point when he is on the road. The number of cars, X, arrived during s seconds follows $\mathsf{Poisson}(\lambda)$ where $\lambda = 3s/60 = s/20$. Thus, the probability that Dave is uninjured is $P(X = 0) = e^{-s/20}$.

58. (a)

$$\begin{aligned} F_Y(c) = P(Y \le c) = P(3X^2 - 4 \le c) &= P(-\sqrt{\frac{c+4}{3}} \le X \le \sqrt{\frac{c+4}{3}}) = F_X(\sqrt{\frac{c+4}{3}}) \\ &= \begin{cases} 0 & c < -1 \\ \frac{1}{3}\sqrt{\frac{c+4}{3}} & -1 \le c < 8 \\ 1 & c \ge 8 \end{cases} \end{aligned}$$

(b) Using the general expectation formula,

$$\begin{aligned} E(Y) &= \int_0^{\infty} (1 - F_Y(c))dc - \int_{-\infty}^{0} F_Y(c)dc \\ &= \int_0^{8} (1 - \frac{1}{3}\sqrt{\frac{c+4}{3}})dc - \int_{-1}^{0} \frac{1}{3}\sqrt{\frac{c+4}{3}}dc \\ &= \frac{10}{3} \end{aligned}$$

59. (a)

$$F_Y(c) = P(Y \le c) = P(e^{sX} \le c) = P(X \le \frac{\ln c}{s}) = \Phi(\frac{\ln c - \mu s}{s\sigma})$$

for $c > 0$.

(b) Using LOTUS, we have

$$\begin{aligned} M(s) = E[e^{sX}] &= \int_{-\infty}^{\infty} \frac{e^{sx}}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-(\mu+s\sigma^2))^2}{2\sigma^2}} e^{\frac{s^2\sigma^2}{2}+\mu s} dx \\ &= e^{\frac{s^2\sigma^2}{2}+\mu s} \end{aligned}$$

(c)

$$\begin{aligned} M'(0) &= (\frac{2s\sigma^2}{2} + \mu)e^{\frac{s^2\sigma^2}{2}+\mu s}|_{s=0} = \mu \\ M''(0) &= (\frac{2s\sigma^2}{2} + \mu)^2 e^{\frac{s^2\sigma^2}{2}+\mu s} + e^{\frac{s^2\sigma^2}{2}+\mu s}\sigma^2|_{s=0} = \mu^2 + \sigma^2 \end{aligned}$$

Alternatively, we can do derivative directly upon $M(s)$. Then,

$$\begin{aligned} M'(0) &= \frac{dE(e^{sX})}{ds}|_{s=0} = E[X] = \mu \\ M''(0) &= \frac{d^2E(e^{sX})}{ds^2}|_{s=0} = E[X^2] = \mu^2 + \sigma^2 \end{aligned}$$