University of Illinois **Fall 2009**

ECE 313: Problem Set 1: Solutions
Sets, Events, Axioms of Probability and Their Consequences

1. **[Subsets of a finite set]**

(a) The subsets of $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ are

Subsets of size 0: $\emptyset$
Subsets of size 1: $\{\omega_1\}$, $\{\omega_2\}$, $\{\omega_3\}$, $\{\omega_4\}$
Subsets of size 2: $\{\omega_1, \omega_2\}$, $\{\omega_1, \omega_3\}$, $\{\omega_1, \omega_4\}$, $\{\omega_2, \omega_3\}$, $\{\omega_2, \omega_4\}$, $\{\omega_3, \omega_4\}$
Subsets of size 3: $\{\omega_1, \omega_2, \omega_3\}$, $\{\omega_1, \omega_2, \omega_4\}$, $\{\omega_1, \omega_3, \omega_4\}$, $\{\omega_2, \omega_3, \omega_4\}$
Subsets of size 4: $\{\omega_1, \omega_2, \omega_3, \omega_4\} = \Omega$

There are $16 = 2^4$ subsets of the set Ω of 4 elements. $15 = 2^4 - 1$ of them are *non-empty* subsets.

(b) OK, OK, geez, some people are never satisfied …

(c) By checking our answers in parts (a) and (b), we see that there is: 1 subset of size 0 and 1 subset of size $4 - 0 = 4$, 4 subsets of size 1 and 4 subsets of size $4 - 1 = 3$, and lastly 6 subsets of size 2 with $4 - 2 = 2$. For general n, whenever we choose a subset of size k, there is a unique subset of size $n - k$ that is left out. That is, for every subset of size k, there is exactly one corresponding, complementary subset of size $n - k$, so the total number of subsets of size k is the same as the total number of subsets of size $n - k$.

(d)
- i. The vector $\{1, 1, \ldots, 1\}$ corresponds to Ω. The vector $\{0, 0, \ldots, 0\}$ corresponds to $\emptyset$. The n-bit vector with each bit flipped with respect to A corresponds to A^c.
- ii. Since $A \cup B$ contains exactly those elements in either A or B, $z_i = x_i \vee y_i$. Similarly, since $A \cap B$ contains exactly those elements in both A and B, $w_i = x_i \wedge y_i$. In purely arithmetical terms, $z_i = x_i + y_i - x_i y_i$ and $w_i = x_i y_i$.
- iii. Each bit of the n-bit vector can take on two values independent of the other bits. Therefore, there are 2^n distinct n-bit vectors. As described in the problem setup, there is a *one-to-one correspondence* between n-bit vectors and subsets of Ω, so there are 2^n subsets.
- iv. The questions is: "What are the n-bit vectors with at least one 1?" Of the 2^n subsets of Ω, there is only one empty subset: $\emptyset$. Consequently, there are $2^n - 1$ non-empty subsets.

(e)
- i. $\binom{4}{0} = 1$, $\binom{4}{1} = 4$, $\binom{4}{2} = 6$, $\binom{4}{3} = 4$, and $\binom{4}{0} = 1$ matching part (a).
- ii. The last term in the series is $\binom{n}{n} = 1$ if n is even and $\binom{n}{n-1} = n$ if n is odd.
- iii. $\binom{n}{1} + \binom{n}{3} + \cdots$. The last term in the series is $\binom{n}{n} = 1$ if n is odd and $\binom{n}{n-1} = n$ if n is even.
- iv. $(1 - x)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k = \binom{n}{0} - \binom{n}{1} x + \binom{n}{2} x^2 - \binom{n}{3} x^3 + \cdots$ and so, setting $x = 1$ and re-arranging, we get that $\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots$ which shows that there are equally many sets of the two types. Since the total number of subsets is $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots$, there are exactly 2^{n-1} subsets whose *size* is an *even* number, and exactly 2^{n-1} subsets whose *size* is an *odd* number.

2. **[A problem on sampling without replacement]**

   (a) There are $\binom{2n}{2}$ choices of two shoes from $2n$ of which only n choices yield a pair. Hence, $P(\text{pair}) = \frac{n}{\binom{2n}{2}} = \frac{n}{2n(2n-1)/1\times 2} = \frac{1}{2n-1}$. More simply, think of choosing the shoes sequentially. Regardless of what the first shoe drawn is, the chance of getting its mate on the second draw is $\frac{1}{2n-1}$. Note that this has value 1 if $n = 1$, which makes perfect sense.

   (b) Any one of the n left shoes can be paired with any one of the n right shoes. So, n^2 of the $\binom{2n}{2}$ choices yield one left shoe and one right shoe in the two drawn, giving that

$$P(\text{one L, one R }) = \frac{n^2}{\binom{2n}{2}} = \frac{n^2}{2n(2n-1)/1\times 2} = \frac{n}{2n-1}.$$

   Again, more simply, regardless of what the first shoe is, the chances of getting a shoe of the opposite footality on the second draw is $\frac{n}{2n-1}$.
   Note that this has value 1 if $n = 1$, which makes perfect sense.

Suppose now that $n \geq 2$ and that you choose 3 shoes at random from the bag.

   (c) Any of the n pairs and any of the other $2n-2$ shoes form a set of 3 shoes. Hence, $P(\text{pair among three}) = \frac{n(2n-2)}{\binom{2n}{3}} = \frac{n(2n-2)}{2n(2n-1)(2n-2)/1\times 2\times 3} = \frac{3}{2n-1}$.
   Note that this has value 1 when $n = 2$, which makes perfect sense.

   (d) $P(\text{three left shoes}) = \frac{\binom{n}{3}}{\binom{2n}{3}} = \frac{n(n-1)(n-2)}{2n(2n-1)(2n-2)} = \frac{n-2}{8n-4}$. Similarly for all three shoes being right shoes. Hence, $P(\text{one L and one R among three}) = 1 - 2\times\frac{n-2}{8n-4} = \frac{3n}{4n-2}$.
   Note that this has value 1 when $n = 2$, which makes perfect sense.

3. **[Unions and Intersections]**

   (a) There are $2^7 = 128$ outcomes with LSB $= 1$, and so $P(A) = \frac{128}{256} = \frac{1}{2}$.

   (b) There are $\binom{8}{5} = \binom{8}{3} = 56$ outcomes in B. Hence $P(B) = \frac{56}{256} = \frac{7}{32}$.

   (c) When the event $A \cap B$ occurs, we know that the 7 more significant bits have exactly 4 ONEs and 3 ZEROes. Hence $|A\cap B| = \binom{7}{4} = \binom{7}{3} = 35$. Thus, $P(A\cap B) = \frac{35}{256}$ and
   $P(A\cup B) = P(A) + P(B) - P(A\cap B) = \frac{128}{256} + \frac{56}{256} - \frac{35}{256} = \frac{149}{256}$, while
   $P(A\oplus B) = P(A) + P(B) - 2\cdot P(A\cap B) = \frac{128}{256} + \frac{56}{256} - 2\cdot\frac{35}{256} = \frac{114}{256} = \frac{57}{128}$.

4. **[New NCAA tournament rules: the Final Five instead of the Final Four!]**

   (a) There are $\binom{5}{2} = 10$ games in this tournament, and $\binom{n}{2} = \frac{n(n-1)}{2}$ games in general.

   (b) i. Two teams cannot possibly have 4-0 records. However, if Team A, say, has a 4-0 record, then we know what happened in 4 games, while the outcomes of the remaining 6 games are arbitrary. Hence, $P(\text{Team A has a 4-0 record}) = \frac{2^6}{2^{10}} = \frac{1}{2^4} = \frac{1}{16}$ and
   $P(\textit{some}\text{ team has a 4-0 record}) = 5\times\frac{1}{16} = \frac{5}{16}$. (What axiom are we using here?)

   ii. The same argument shows that $P(\textit{some}$ team has a 0-4 record$) = \dfrac{5}{16}$.
   iii. If Team A wins all four of its games *and* Team B loses all four of its games, then we know what happened in 7 games (why not 8?), and hence we get that
   $P(\text{Team A has 4-0 record; Team B has a 0-4 record}) = \dfrac{2^3}{2^{10}} = \dfrac{1}{2^7} = \dfrac{1}{128}$, and
   $P(\text{some team has 4-0 record; some other team has a 0-4 record}) = 20 \times \dfrac{1}{128} = \dfrac{5}{32}$.
   iv. The remaining teams have lost one game (against the 4-0 team) and won another game (against the 0-4 team), and will have identical 2-2 records if each wins one game and loses one game among the three games that these teams play against one another (e.g. A beats B who beats C who beats A: basketball is not necessarily a transitive game!). Since only 2 of the 8 outcomes of these three games give 2-2 records for all three teams, we get that
   $P(\text{one team is 4-0; another is 0-4; rest are 2-2}) = \dfrac{5}{32} \times \dfrac{2}{8} = \dfrac{5}{128}$.

5. **[Drill in working with subsets]**

   $P(A \cup (B^c \cup C^c)^c) = P(A \cup (B \cap C))$ by DeMorgan's theorem.

   (a) $B \cap C = \emptyset$ and hence $P(A \cup (B \cap C)) = P(A \cup \emptyset) = P(A) = 1/3$.

   (b) $P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap (B \cap C))$
   $= 4P(A \cap B \cap C) + 2P(A \cap B \cap C) - P(A \cap B \cap C) = 5P(A \cap B \cap C) = 5/8$.

   (c) $P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap (B \cap C)) = 1/2 + 1/3 - 0 = 5/6$. Why?

   (d) $(A \cup (B^c \cup C^c)^c)^c = A^c \cap (B^c \cup C^c)$. Hence, $P(A \cup (B^c \cup C^c)^c) = 1 - 0.6 = 0.4$.