Let h(x) denote a differentiable function of x. Then, for small values of λ, we can approximate $h(x+\lambda) - h(x)$ as follows:

$$h(x+\lambda) - h(x) \approx \lambda \frac{d}{dx} h(x).$$

We apply this idea to finding the derivative of an integral. In the remainder of this discussion, we shall assume that all functions that we choose to integrate are indeed integrable, and that all the functions that we choose to differentiate are indeed differentiable.

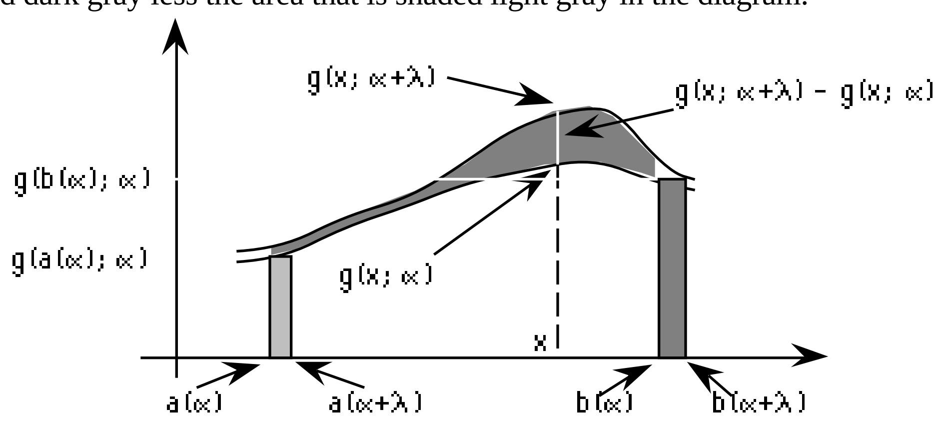
The basic result is as follows. Let $g(x;\alpha)$ denote a function of x in which α is a parameter, and let $a(\alpha)$ and $b(\alpha)$ denote two functions of α. Let

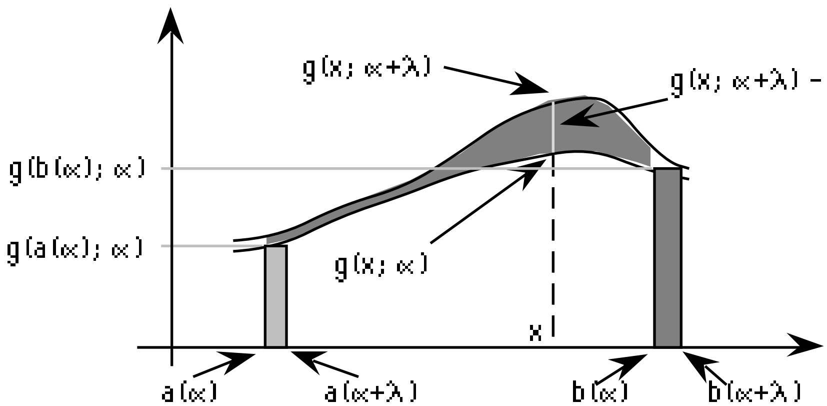
$$G(\alpha) = \int_{a(\alpha)}^{b(\alpha)} g(x;\alpha)dx$$

where the integrand as well as both the upper and lower limits are dependent on α. Then, the derivative of $G(\alpha)$ with respect to α is given by

$$\frac{dG(\alpha)}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial g(x;\alpha)}{\partial\alpha} dx + g(b(\alpha);\alpha)\frac{db(\alpha)}{d\alpha} - g(a(\alpha);\alpha)\frac{da(\alpha)}{d\alpha} \tag{1}$$

That this result is plausible can be shown using a diagram such as that shown below. For small values of λ, $G(\alpha+\lambda) - G(\alpha) \approx \lambda\frac{dG(\alpha)}{d\alpha}$. But, since $G(\alpha)$ is the area under the curve $g(x;\alpha)$ from $a(\alpha)$ to $b(\alpha)$ while $G(\alpha+\lambda)$ is the area under $g(x;\alpha+\lambda)$ from $a(\alpha+\lambda)$ to $b(\alpha+\lambda)$, we see that the quantity $G(\alpha+\lambda) - G(\alpha)$ is *approximately* the area that is shaded dark gray less the area that is shaded light gray in the diagram.





The areas of the two rectangular strips are very obviously given by

$$g(a(\alpha);\alpha)\times[a(\alpha+\lambda) - a(\alpha)] \approx g(a(\alpha);\alpha)\times\lambda\times\frac{da(\alpha)}{d\alpha}$$

and

$$g(b(\alpha);\alpha)\times[b(\alpha+\lambda)-a(\alpha)] \approx g(b(\alpha);\alpha)\times\lambda\times\frac{db(\alpha)}{d\alpha}.$$

On the other hand, the area between the two curves is just the integral of the function $g(x;\alpha+\lambda) - g(x;\alpha) \approx \lambda\frac{\partial g(x;\alpha)}{\partial\alpha}$ from approximately $a(\alpha)$ to approximately $b(\alpha)$. Putting all these results together gives

$$G(\alpha+\lambda) - G(\alpha) \approx \lambda\frac{dG(\alpha)}{d\alpha} \approx \lambda\left[\int_{a(\alpha)}^{b(\alpha)} \frac{\partial g(x;\alpha)}{\partial\alpha}\,dx + g(b(\alpha);\alpha)\frac{db(\alpha)}{d\alpha} - g(a(\alpha);\alpha)\frac{da(\alpha)}{d\alpha}\right]$$

and (1) follows from this. A more formal ε-δ proof can be constructed along these lines by interested parties.

Two special cases of this result occur often enough to be noted separately. If the limits do not depend on α, that is, the upper and lower limits are just the constants a and b, then we have that

$$\frac{d}{d\alpha}\int_a^b g(x;\alpha)dx = \int_a^b \frac{\partial g(x;\alpha)}{\partial\alpha}\,dx \tag{2}$$

On the other hand, if the integrand does not depend on α but the limits do, then

$$\frac{d}{d\alpha}\int_{a(\alpha)}^{b(\alpha)} g(x)\,dx = g(b(\alpha);\alpha)\frac{db(\alpha)}{d\alpha} - g(a(\alpha);\alpha)\frac{da(\alpha)}{d\alpha} \tag{3}$$

As an example of the application of these special cases, consider that if $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$, then

$$F_{\mathbf{Z}}(\alpha) = P\{\mathbf{X}+\mathbf{Y} < \alpha\} = \int_{u=-\infty}^{\infty}\int_{v=-\infty}^{\alpha-u} f_{\mathbf{X},\mathbf{Y}}(u,v)\,dv\,du.$$

Hence, $f_{\mathbf{Z}}(\alpha) = \frac{d}{d\alpha}P\{\mathbf{X}+\mathbf{Y} < \alpha\} = \int_{u=-\infty}^{\infty}\frac{d}{d\alpha}\int_{v=-\infty}^{\alpha-u} f_{\mathbf{X},\mathbf{Y}}(u,v)\,dv\,du = \int_{u=-\infty}^{\infty} f_{\mathbf{X},\mathbf{Y}}(u,\alpha-u)\,du$

where we have used (2) to differentiate the outer integral and (3) to differentiate the inner integral.