University of Illinois **Spring 2009**

ECE 313: Problem Set 11: Solutions
Joint Distributions of Random Variables

1. **[Low-pass filtering a signal]**

   $Y(f) = H(f)X(f) = \text{rect}(f/2)\exp(-\pi f^2)$, and $y(0) = \int_{-\infty}^{\infty} Y(f)\,df = \int_{-1}^{1} \exp(-\pi f^2)\,df$. But, the *integrand* is the Gaussian pdf $\mathcal{N}(0,(2\pi)^{-1})$.
   Hence, $y(0) = \Phi(\sqrt{2\pi}) - \Phi(-\sqrt{2\pi}) = 2\Phi(\sqrt{2\pi}) - 1 \approx 0.9826$

2. **[Joint pmfs]**

   (a) The marginal pmfs $p_{\mathbb{X}}(u)$ and $p_{\mathbb{Y}}(v)$ are column and row sums as shown in the table below.

$u \rightarrow$ $v \downarrow$	0	1	3	5	Row sum
4	0	1/12	1/6	1/12	1/3
3	1/6	1/12	0	1/12	1/3
-1	1/12	1/6	1/12	0	1/3
Column sum	1/4	1/3	1/4	1/6	1

   (b) The eyeball test tells us that $\mathbb{X}$ and $\mathbb{Y}$ are *dependent* random variables. Less optically, $p_{\mathbb{X},\mathbb{Y}}(0,4) = 0 \neq p_{\mathbb{X}}(0)p_{\mathbb{Y}}(4) = \frac{1}{4}\times\frac{1}{3} = \frac{1}{12}$ and so $\mathbb{X}$ and $\mathbb{Y}$ are not independent random variables.

   (c) $P\{\mathbb{X} \le \mathbb{Y}\} = p_{\mathbb{X},\mathbb{Y}}(0,3) + p_{\mathbb{X},\mathbb{Y}}(0,4) + p_{\mathbb{X},\mathbb{Y}}(1,3) + p_{\mathbb{X},\mathbb{Y}}(1,4) + p_{\mathbb{X},\mathbb{Y}}(3,4) = \dfrac{1}{2}$.

   $P\{\mathbb{X}+\mathbb{Y} \le 4\} = p_{\mathbb{X}}(0) + p_{\mathbb{X},\mathbb{Y}}(1,3) + p_{\mathbb{X},\mathbb{Y}}(1,-1) + p_{\mathbb{X},\mathbb{Y}}(3,-1) + p_{\mathbb{X},\mathbb{Y}}(5,-1) = \dfrac{7}{12}$.

   (d) $p_{\mathbb{X}|\mathbb{Y}}(u|3) = \dfrac{p_{\mathbb{X},\mathbb{Y}}(u,3)}{p_{\mathbb{Y}}(3)} = 1/2, 1/4, 1/4$ respectively for $u = 0,1,5$.

   $\mathsf{E}[\mathbb{X} \mid \mathbb{Y}=3] = 0\times\dfrac{1}{2} + 1\times\dfrac{1}{4} + 5\times\dfrac{1}{4} = \dfrac{3}{2}$.

   $\mathsf{var}(\mathbb{X} \mid \mathbb{Y}=3) = \mathsf{E}[\mathbb{X}^2 \mid \mathbb{Y}=3] - (\mathsf{E}[\mathbb{X} \mid \mathbb{Y}=3])^2 = 1\times\dfrac{1}{4} + 5^2\times\dfrac{1}{4} - \left(\dfrac{3}{2}\right)^2 = \dfrac{26}{4} - \dfrac{9}{4} = \dfrac{17}{4}$.

3. **[Drill problem on jointly continuous random variables I]**

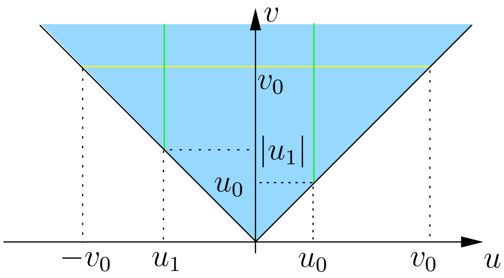
   The pdf is nonzero over the shaded region in the left-hand figure shown on the next page.

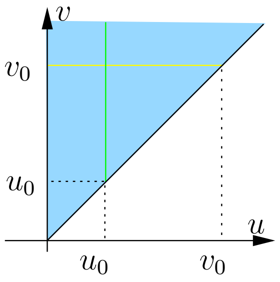
   (a) , (b) From the figure, we get that for any $v_0 > 0$,

$$f_{\mathbb{Y}}(v_0) = \int_{u=-v_0}^{u=+v_0} c(v^2-u^2)\exp(-v)\,du = c\left[v^2u - \frac{u^3}{3}\right]\exp(-v)\Big|_{u=-v_0}^{u=+v_0} = c\left(\frac{4}{3}\right)v_0^3\exp(-v_0)$$

   which is of the form of a *gamma* pdf with parameters $(4,1)$. Thus, $4c/3 = 1/\Gamma(4) = 1/3 \Rightarrow c = 1/8$. On the other hand, for any $u_0 > 0$, $f_{\mathbb{X}}(u_0) = \int_{v=u_0}^{\infty} \frac{1}{8}(v^2-u^2)\exp(-v)\,dv = \frac{1}{4}(1+u_0)\exp(-u_0)$, while for $u_1 < 0$, the limits are $v = |u_1|$ and ∞ giving $f_{\mathbb{X}}(u_1) = \frac{1}{4}(1+|u_1|)\exp(-|u_1|)$. Consequently, $f_{\mathbb{X}}(u) = \frac{1}{4}(1+|u|)\exp(-|u|), -\infty < u < \infty$.

   (c) The pdf of $\mathbb{X}$ is an even function of u and $\int_0^{\infty} u\cdot f_{\mathbb{X}}(u)\,du$ is finite. Hence, $\mathsf{E}[\mathbb{X}] = 0$.





4. **[Drill problem on jointly continuous random variables II]**

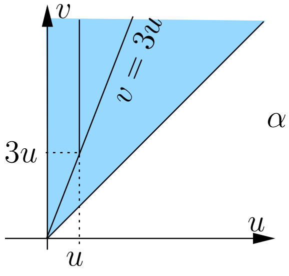
   (a) The joint pdf is nonzero on the shaded region shown in the right-hand figure above.

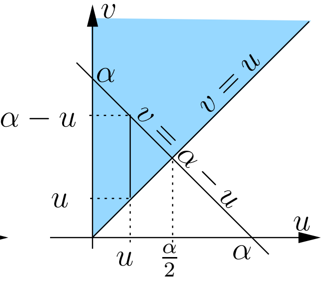
   (b) For any $u_0 > 0$, $f_{\mathbb{X}}(u_0) = \int_{v=u_0}^{\infty} 2\exp(-u_0 - v)\,dv = 2\exp(-2u_0)$.

   For any $v_0 > 0$, $f_{\mathbb{Y}}(v_0) = \int_{u=0}^{V_0} 2\exp(-u - v_0)\,du = 2\exp(-v_0) - 2\exp(-2v_0)$. Consequently, $f_{\mathbb{X}}(u) = \begin{cases} 2\exp(-2u), & u > 0, \\ 0, & \text{elsewhere,} \end{cases}$ and $f_{\mathbb{X}}(u) = \begin{cases} 2\exp(-v) - 2\exp(-2v), & v > 0, \\ 0, & \text{elsewhere.} \end{cases}$

   (c) No, the eyeball test says that the random variables are dependent. Less optically, $f_{\mathbb{X},\mathbb{Y}}(u,v) = 0 \neq f_{\mathbb{X}}(u)f_{\mathbb{Y}}(v)$ for any u and v such that $0 < v < u$.

   (d) $P\{\mathbb{Y} > 3\mathbb{X}\} = \int_{u=0}^{\infty}\int_{v=3u}^{\infty} 2\exp(-u-v)\,dv\,du = \int_{u=0}^{\infty} 2\exp(-4u)\,du = \frac{1}{2}$.
   See the left-hand figure below.





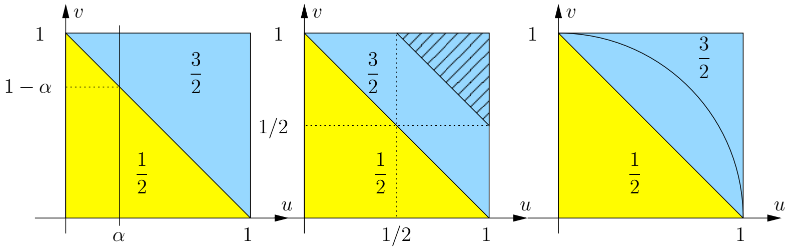
   (e) See the right-hand figure above. For $\alpha > 0$, we have

   $$P\{\mathbb{X}+\mathbb{Y} \leq \alpha\} = \int_{u=0}^{\alpha/2}\int_{v=u}^{\alpha-u} 2e^{-u-v}\,dv\,du = \int_{u=0}^{\alpha/2} 2e^{-u}[e^{(-u)} - e^{(-\alpha+u)}]\,du = 1-(1+\alpha)\exp(-\alpha).$$

   (f) $f_{\mathbb{Z}}(\alpha) = \frac{d}{d\alpha}F_{\mathbb{Z}}(\alpha) = \frac{d}{d\alpha}P\{\mathbb{X}+\mathbb{Y} \leq \alpha\} = \frac{d}{d\alpha}1 - (1+\alpha)\exp(-\alpha) = \alpha\exp(-\alpha)$ for $\alpha > 0$ and $f_{\mathbb{Z}}(\alpha) = 0$ for $\alpha < 0$. This is a *gamma* pdf with parameters $(2, 1)$.

5. **[Drill problem on jointly continuous random variables III]**

   (a) The joint pdf is illustrated in the left-hand figure below.



From the left-hand figure above, $f_{\mathbb{X}}(\alpha) = 0$ for $\alpha < 0$ or $\alpha > 1$, while for any $\alpha, 0 \le \alpha \le 1$,

$$f_{\mathbb{X}}(\alpha) = \int_{-\infty}^{\infty} f_{\mathbb{X},\mathbb{Y}}(\alpha, v)\,dv = \int_{0}^{1-\alpha} \frac{1}{2}\,dv + \int_{1-\alpha}^{1} \frac{3}{2}\,dv = \frac{1}{2}(1-\alpha) + \frac{3}{2}\alpha = \frac{1}{2} + \alpha.$$

(b) When the pdf has constant value over a region, we can find the probability that the random point lies in that region by finding the area of the region and multiplying by the pdf value. Thus, $P\{\mathbb{X} + \mathbb{Y} \le 3/2\} = 1 - P\{\mathbb{X} + \mathbb{Y} > 3/2\} = 1 - \frac{3}{2} \times \left[\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right] = \frac{13}{16}$
and $P\{\mathbb{X}^2 + \mathbb{Y}^2 \ge 1\} = \frac{3}{2} \times \left[1 - \frac{\pi}{4}\right] = \frac{3}{2} - \frac{3\pi}{8}$.

6. **[Average of n independent random variables]**

(a) From the transform pair $\exp(-t^2/2\sigma^2) \leftrightarrow \sigma\sqrt{2\pi}\exp(-\sigma^2\omega^2/2)$ in the ECE 210 book, we readily get $\frac{1}{\sigma\sqrt{2\pi}}\exp(-t^2/2\sigma^2) \leftrightarrow \exp(-\sigma^2\omega^2/2)$.

(b) From the time shift theorem $f(t-\tau) \leftrightarrow F(\omega)\exp(-j\omega\tau)$, we get the transform pair $\frac{1}{\sigma\sqrt{2\pi}}\exp(-(t-\mu)^2/2\sigma^2) \leftrightarrow \exp(-\sigma^2\omega^2/2)\exp(-j\omega\mu)$.

(c) Convolution in the time domain is multiplication in the frequency domain. Hence, the Fourier transform of the pdf of $\mathbb{Z}$ is the product $[\exp(-\sigma^2\omega^2/2)\exp(-j\omega\mu)]^n$ of the Fourier transforms of the pdfs of the $\mathbb{X}_i$. Note that the product simplifies to $\exp(-n\sigma^2\omega^2/2)\exp(-j\omega n\mu)$. Notice also that if the pdfs had different means μ_i and variances σ_i^2, the transform of the pdf of $\mathbb{Z}$ would have had $\sum \sigma_i^2$ and $\sum \mu_i$ in place of $n\sigma^2$ and $n\mu$ respectively.

(d) Writing $\beta^2 = n\sigma^2$ and $\nu = n\mu$, we see from the result of part (b) that $\mathbb{Z}$ is a Gaussian random variable with mean $\nu = n\mu$ and variance $\beta^2 = n\sigma^2$. The additional comments in the solution of part (c) are thus an informal proof of Proposition 3.2 in Chapter 6 of the textbook.

(e) Since $\mathbb{Z}$ is a $\mathcal{N}(n\mu, n\sigma^2)$ random variable, it follows readily that $\mathbb{W} = \mathbb{Z}/n$ is a $\mathcal{N}(\mu, \sigma^2/n)$ random variable. Thus, the average of n independent Gaussian random variables is a Gaussian random variable with the same mean and a much smaller variance.

(f) From the transform pair $\frac{a^2}{a^2+t^2} \leftrightarrow \pi a \exp(-a|\omega|)$ in the ECE 210 book, we readily get $\frac{1}{1+t^2} \leftrightarrow \pi\exp(-|\omega|)$.

(g) $\frac{1}{\pi(1+t^2)} \leftrightarrow \exp(-|\omega|) \Rightarrow \frac{1}{\pi(1+(t-\mu)^2)} \leftrightarrow \exp(-|\omega|)\exp(-j\mu\omega)$.

(h) The Fourier transform of the pdf of $\mathbb{Z}$ is $[\exp(-|\omega|)\exp(-j\mu\omega)]^n = \exp(-n|\omega|)\exp(-jn\mu\omega)$. Since $\frac{n}{\pi(n^2+t^2)} \leftrightarrow \exp(-n|\omega|)$, the pdf of $\mathbb{Z}$ is $f_{\mathbb{Z}}(t) = \frac{n}{\pi(n^2+(t-n\mu)^2)}, \quad -\infty < t < \infty$.

(i) $f_{\mathbb{W}}(t) = nf_{\mathbb{Z}}(nt) = \frac{1}{\pi(1+(t-\mu)^2)}, -\infty < t < \infty$, which is the same as the pdf of the $\mathbb{X}_i$'s that we started with!

Note that Fourier transforms are found in probability theory under the name *characteristic functions* but are not discussed in the textbook.

7. **[Dividing a Poisson stream into two streams]**

(a) If server A has been sent m packets, that is, if $\mathbb{Y} = m$, then it must be that *at least* m packets were received by the router, that is, $\mathbb{X} \ge m$. Hence, we have that for $m \ge 0$,

$$p_{\mathbb{Y}}(m) = \sum_{n=m}^{\infty} p_{\mathbb{Y}|\mathbb{X}}(m \mid n)p_{\mathbb{X}}(n) = \sum_{n=m}^{\infty} \frac{n!p^m(1-p)^{n-m}}{m!(n-m)!} \cdot \frac{(\lambda T)^n\exp(-\lambda T)}{n!}$$
$$= \frac{(\lambda T)^m}{m!}\exp(-\lambda T)\sum_{n=m}^{\infty}\frac{[(1-p)\lambda T]^{n-m}}{(n-m)!} = \frac{(\lambda T)^m}{m!}\exp(-\lambda T)\sum_{i=0}^{\infty}\frac{[(1-p)\lambda T]^i}{i!}$$
$$= \frac{(\lambda T)^m}{m!}\exp(-\lambda T)\exp((1-p)\lambda T) = \frac{(\lambda T)^m}{m!}\exp(-\lambda pT),$$ that is, $\mathbb{Y}$ is a Poisson random variable

with parameter $(\lambda p)T$. In fact, the streams of arrivals at the two servers A and B are Poisson processes with arrival rates λp and $\lambda(1-p)$ respectively. This result is called *Poisson splitting*. Note that if the packets are unaddressed, and the router simply sends alternate packets to A and B (so as to balance the load on the servers), then the streams of arrivals at the servers are *not* Poisson processes with arrival rate $\lambda/2$ each.

(b) When $\mathbb{Y} = m$ is observed, it must be that $\mathbb{X} \geq m$. Hence, for $n \geq m \geq 0$,

$$p_{\mathbb{X}|\mathbb{Y}}(n \mid m) = \frac{p_{\mathbb{Y}|\mathbb{X}}(m \mid n)p_{\mathbb{X}}(m)}{p_{\mathbb{Y}}(n)} = \frac{\binom{n}{m}p^m(1-p)^{n-m} \cdot (\lambda T)^n \exp(-\lambda T)/n!}{(\lambda p T)^m \cdot \exp(-\lambda p T)/m!}$$

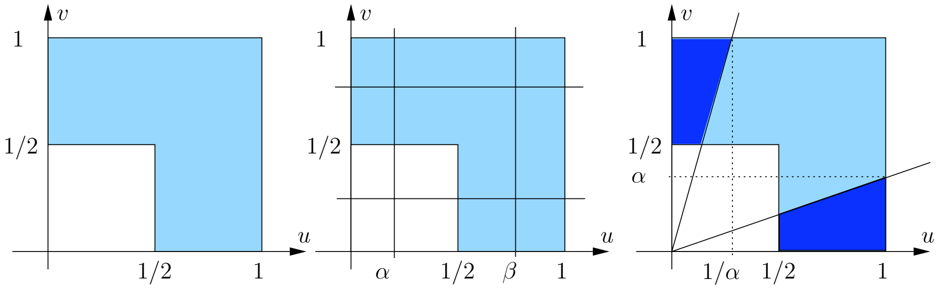
$$= \frac{(\lambda(1-p)T)^{n-m}\exp(-\lambda(1-p)T)}{(n-m)!}.$$

This is a *displaced* Poisson pmf: the conditional pmf of $\mathbb{X}$ is that of the random variable $m+\mathbb{Z}$ where $\mathbb{Z}$ is a Poisson random variable with parameter $\lambda(1-p)T$. Note that $m+\mathbb{Z}$ takes on values $m, m+1, \ldots$

(c) $\mathsf{E}[\mathbb{X} \mid \mathbb{Y} = m] = \mathsf{E}[m + \mathbb{Z}] = m + \lambda(1-p)T$.

8. [**A piece of cake? or a sheet cake with a piece missing?**]

(a) The pdf is nonzero over the shaded region in the left-hand figure shown below.



(b) Since the area of the shaded region is $\dfrac{3}{4}$, we have that $c = \dfrac{4}{3}$.

(c) $f_{\mathbb{Y}}(v) = \displaystyle\int_{-\infty}^{\infty} f_{\mathbb{X},\mathbb{Y}}(u,v)\,du = \begin{cases} \int_{1/2}^{1} \frac{4}{3}\,du = \frac{2}{3}, & 0 < v \leq \frac{1}{2}, \\ \int_{0}^{1} \frac{4}{3}\,du = \frac{4}{3}, & \frac{1}{2} < v < 1, \end{cases}$ and $f_{\mathbb{Y}}(v) = 0$ otherwise. Use the horizontal lines in the middle figure above if you have difficulty figuring out where the limits came from. By symmetry, $\mathbb{X}$ has the same marginal pdf as $\mathbb{Y}$.

(d) $\mathsf{E}[\mathbb{Y}] = \displaystyle\int_{-\infty}^{\infty} v \cdot f_{\mathbb{X}}(v)\,dv = \int_{0}^{1/2} v \cdot \frac{2}{3}\,dv + \int_{1/2}^{1} v \cdot \frac{4}{3}\,dv = \frac{7}{12}.$

(e) **Comment on parts (e) and (f):** $f_{\mathbb{Y}|\mathbb{X}}(v \mid \alpha) = \dfrac{f_{\mathbb{X},\mathbb{Y}}(\alpha, v)}{f_{\mathbb{X}}(\alpha)}$ provided that $f_{\mathbb{X}}(\alpha) > 0$. More graphically, given $\mathbb{X} = \alpha$, the conditional pdf $f_{\mathbb{Y}|\mathbb{X}}(v|\alpha)$ is the *cross-section* of the pdf surface at $u = \alpha$ unitized to have area 1. As should be obvious by looking at the vertical lines in the middle figure above, the cross-section is a rectangle of base $1/2$ or 1 depending on the value of α.
For $0 < \alpha < \frac{1}{2}$: $f_{\mathbb{Y}|\mathbb{X}}(v|\alpha) \sim \mathsf{Uniform}(\frac{1}{2}, 1) \Rightarrow \mathsf{E}[\mathbb{Y}|\mathbb{X} = \alpha] = \frac{3}{4}$, $\mathsf{var}[\mathbb{Y}|\mathbb{X} = \alpha] = \frac{1}{48}$.

(f) For $\frac{1}{2} < \beta < 1$: $f_{\mathbb{Y}|\mathbb{X}}(v|\beta) \sim \mathsf{Uniform}(0, 1) \Rightarrow \mathsf{E}[\mathbb{Y}|\mathbb{X} = \beta] = \frac{1}{2}$, $\mathsf{var}[\mathbb{Y}|\mathbb{X} = \beta] = \frac{1}{12}$.

(g) $\mathbb{Y} \leq \alpha\mathbb{X}$ if the random point lies in the deep-shaded region in the lower right corner of the rightmost figure above. The area is $\frac{\alpha}{2} - \frac{\alpha}{8} = \frac{3\alpha}{8}$ and hence $P\{\mathbb{Y} \leq \alpha\mathbb{X}\} = \frac{4}{3} \times \frac{3\alpha}{8} = \frac{\alpha}{2}$.

(h) $\mathbb{Y} > \alpha\mathbb{X}$ if the random point lies in the deep-shaded region in the upper left corner of the rightmost figure above. The area is $\frac{1}{2\alpha} - \frac{1}{8\alpha} = \frac{3}{8\alpha}$ and hence $P\{\mathbb{Y} \leq \alpha\mathbb{X}\} = 1 - \frac{4}{3} \times \frac{3}{8\alpha} = 1 - \frac{1}{2\alpha}$.

(i) $P\{\mathbb{Z} \leq \alpha\} = P\{\mathbb{Y} \leq \alpha\mathbb{X}\} = \begin{cases} \frac{\alpha}{2}, & 0 < \alpha \leq 1, \\ 1 - \frac{1}{2\alpha}, & 1 < \alpha < \infty. \end{cases}$

(j) $f_{\mathbb{Z}}(\alpha) = \frac{d}{d\alpha}F_{\mathbb{Z}}(\alpha) = \frac{d}{d\alpha}P\{\mathbb{Z} \leq \alpha\} = \begin{cases} \frac{1}{2}, & 0 < \alpha \leq 1, \\ \frac{1}{2\alpha^2}, & 1 < \alpha < \infty, \\ 0, & \text{elsewhere.} \end{cases}$