ECE 361: Digital Communication
Lecture 10: Capacity of the Continuous time AWGN Channel

Introduction

In the penultimate lecture we saw the culmination of our study of reliable communication on the discrete time AWGN channel. We concluded that there is a threshold called capacity below which we are guaranteed arbitrarily reliable communication and above which all communication is hopelessly unreliable. But the real world is analog and in the last lecture we saw in detail the engineering way to connect the *continuous time* AWGN channel to the discrete time one. In this lecture we will connect these two story lines into a final statement: we will derive a formula for the capacity of the continuous time AWGN channel. This is the largest rate of reliable communication (as measured in bits/second) and depends only on the two key physical resources: bandwidth and power. We will see the utility of this formula by getting a feel for how the capacity changes as a function of the two physical has more impact on the capacity

The Continuous Time AWGN Channel

The channel is, naturally enough,

$$y(t) = x(t) + w(t), \quad t > 0. \tag{1}$$

The power constraint of $\bar{P}$ Watts on the transmit signal says that

$$\lim_{N\to\infty} \frac{1}{NT} \int_0^{NT} (x(t))^2 \; dt \quad \leq \quad \bar{P}. \tag{2}$$

The (two-sided) bandwidth constraint of W says that much of the energy in the transmit signal is contained within the spectral band $\left[-\frac{W}{2}, \frac{W}{2}\right]$.

We would like to connect this to the discrete time AWGN channel:

$$y[m] = x[m] + w[m], \quad m \geq 1. \tag{3}$$

This channel came with the discrete-time power constraint:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} (x(m])^2 \leq P \quad \forall N. \tag{4}$$

We have already seen that there are W channel uses per second in the continuous time channel if we constrain the bandwidth of the analog transmit voltage waveform to W Hz. So, this fixes the sampling rate to be W and thus unit time in the discrete time channel corresponds to $\frac{1}{W}$ seconds.

To complete the connection we need to:

1. connect the two power constraints $\bar{P}$ and P;

2. find an appropriate model for the continuous time noise $w(t)$ and connect it to the variance of the additive noise $w[m]$.

We do these two steps next.

Connecting the Power Constraints

The continuous time transmit waveform is the related to the discrete sequence of transmit voltages through the DAC operation (cf. Lecture 9):

$$x(t) = \sum_{m>0} x[m] g(t - mT). \tag{5}$$

Now we have

$$\begin{aligned}
\frac{1}{NT}\int_0^{NT} (x(t))^2 \, dt &= \frac{1}{NT}\int_0^{NT} \left(\sum_{m>0} x[m] g(t-mT)\right)^2 dt && (6)\\
&= \frac{1}{NT}\sum_{m_1,m_2>0} x[m_1]x[m_2]\left(\int_0^{NT} g(t-m_1T)g(t-m_2T)\right) dt && (7)\\
&= \frac{1}{NT}\sum_{m_1=m_2=m>0} (x[m])^2 \int_0^{NT} (g(t-mT))^2 \, dt && (8)\\
&\quad + \frac{1}{NT}\sum_{m_1\neq m_2>0} x[m_1]x[m_2]\left(\int_0^{NT} g(t-m_1T)g(t-m_2T)\right) dt && (9)
\end{aligned}$$

Consider the first term of the RHS above:

$$\frac{1}{NT}\sum_{m_1=m_2=m>0} (x[m])^2 \int_0^{NT} (g(t-mT))^2 \, dt. \tag{10}$$

From Lecture 9 we know that the pulse $g(\cdot)$ has a finite spread of T_p; in our notation, the pulse is nonzero mostly over the range $[-t_0, T_p + t_0]$. Further, we have seen that typically T_p a few multiples of T. From this, we can make the following two observations:

1. the summation index m in Equation (10) spans from 1 to

$$N + \frac{T_p - t_0}{T} \approx N, \tag{11}$$

   when N is large enough.

2. Next, the integral

$$\frac{1}{T}\int_0^{NT} (g(t-mT))^2 \, dt \tag{12}$$

   is more or less constant for each m in the range from 1 to N (except perhaps for a few values at the boundary.

We can now combine these two observations to conclude that the term in Equation (10) is approximately the same as

$$\frac{c_p}{N}\sum_{m=1}^{N} (x(m])^2, \tag{13}$$

which we see, by comparing with Equation (4), is directly proportional to the discrete time power consumed.

To complete the connection, we still need to account for the second term in the RHS of Equation (9):

$$\frac{1}{NT} \sum_{m_1 \neq m_2 > 0} x[m_1]x[m_2] \left(\int_0^{NT} g(t - m_1 T) g(t - m_2 T) \, dt \right). \tag{14}$$

Fortunately, for most pulses of interest this quantity is zero. Specifically, this statement is true for the three exemplar pulses of Lecture 9: the `sinc`, `rect` and `raised cosine` pulses. This is verified in a homework exercise. In practice, the term in Equation (14) is kept reasonably small and we can ignore its effect on the summation in Equation (9). Thus,

$$\frac{1}{NT} \int_0^{NT} (x(t))^2 \, dt \quad \approx \quad \frac{c_p}{N} \sum_{m=1}^{N} (x(m])^2. \tag{15}$$

Combining this with Equations (2) and (4) we arrive at

$$\bar{P} = c_p P. \tag{16}$$

The constant c_p is a design choice that depends on the energy in the DAC pulse. For notational simplicity we will simply consider it to be unity. This allows us to map the continuous time power constant of $\bar{P}$ directly into the P, the discrete time power constraint.

Analog Noise Models and Bandwidth

Consider the continuous time noise $w(t)$ it's (random) Fourier transform. Since noise waveforms are *power* signals, i.e.,

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (w(t))^2 \, dt \quad = \quad \sigma^2 > 0 \tag{17}$$

the Fourier transform is not well defined. To avoid this problem we could consider the *time-restricted* noise waveform

$$w_T(t) \stackrel{\text{def}}{=} \begin{cases} w(t) & -T \leq t \leq T \\ 0 & \text{else} \end{cases} \tag{18}$$

which is an *energy* signal. We denote the Fourier transform of $w_T(t)$ by $W_T(f)$. The *average* variance of the Fourier transform of the time-restricted noise in the limit of no restriction is called the *power spectral density*:

$$\text{PSD}_w(f) \stackrel{\text{def}}{=} \lim_{T \to \infty} \frac{1}{2T} \text{Var}\left(W_T(f)\right). \tag{19}$$

Based on measurements of additive noise, a common model for the power spectral density is that it is constant, denoted by $\frac{N_0}{2}$, measured in Watts/Hz. Furthermore this model holds over a very wide range of frequencies of interest to communication engineers: practical measurement data suggests a value of about 10^{-14} Watts/Hz for N_0.

The corresponding statistical model of $w(t)$ is called *white Gaussian noise*. In modeling the discrete time noise (cf. Lecture 2) we used the term "white" to denote statistical independence of noise over different time samples. Here the term "white" is being used to denote statistical independence of the continuous time noise $w(t)$ over different frequencies.

This model immediately implies the following strategy: consider the received signal $y(t)$. The transmit signal is known to be bandlimited to $x(t)$ and the additive noise $w(t)$ is independent over different frequencies. So, without loss of generality:

we can *filter* the received waveform $y(t)$ so that it is bandlimited to W as well.

In practice, the received waveform is always filtered to contain it to within the same spectral band as that of the transmit waveform. Filtering the received waveform is tantamount to filtering the noise waveform alone (since the transmit waveform is anyway in the same band as that allowed by the filter). With a (double sided) bandwidth of W, the total area under the power spectral density of this filtered noise is

$$\int_{-\frac{W}{2}}^{\frac{W}{2}} \mathrm{PSD}(f)\, df = \frac{N_0 W}{2}. \tag{20}$$

It turns out that the variance of the noise sample $w[m]$, at time sample m, is *exactly equal* to the expression in Equation (20)! This calculation is explored in a bonus homework exercise. So, we conclude that the variance of the noise sample increases in direct proportion to the bandwidth W.

Capacity

We can now put together these observations into our earlier discussion of the capacity of the discrete time AWGN channel,

$$\frac{1}{2}\log_2\left(1+\frac{P}{\sigma^2}\right) \quad \text{bits/unit time,} \tag{21}$$

to arrive at the capacity of the continuous time AWGN channel:

$$C = \frac{W}{2}\log_2\left(1+\frac{2\bar{P}}{N_0 W}\right) \quad \text{bits/s.} \tag{22}$$

Now we can see how the capacity depends on the bandwidth W. Surely the capacity can only increase as W increases (one can always ignore the extra bandwidth). One can directly show that the capacity is a *concave* function of the bandwidth W (this is explored in an exercise). Figure 1 plots the variation of the capacity as a function of bandwidth for an exemplar value of SNR per Hz.

Two important implications follow:

- When the bandwidth is small, the capacity is very sensitive to changes in bandwidth: this is because the SNR per Hz is quite large and then the capacity is pretty much linearly related to the bandwidth. This is called the *bandwidth limited* regime.

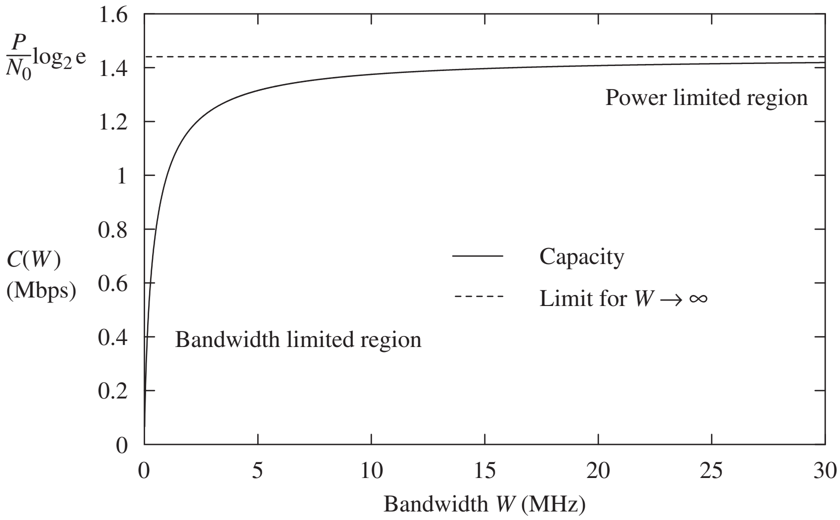

Figure 1: Capacity as a function of bandwidth for SNR per Hz $2P/N_0 = 10^6$.

- When the bandwidth is large, the SNR per Hz is small and

$$\frac{W}{2}\log_2\left(1+\frac{2\bar{P}}{N_0W}\right) \approx \frac{W}{2}\left(\frac{2\bar{P}}{N_0W}\right)\log_2 e \tag{23}$$

$$= \frac{\bar{P}}{N_0}\log_2 e. \tag{24}$$

  In this regime, the capacity is proportional to the total power $\bar{P}$ received over the whole band. It is insensitive to the bandwidth and increasing the bandwidth has only a small impact on capacity. On the other hand, the capacity is now linear in the received power and increasing power does have a significant effect. This is called the *power limited* regime.

As W increases, the capacity increases monotonically and reaches the asymptotic limit

$$C_\infty = \frac{\bar{P}}{N_0}\log_2 e \quad \text{bits/s.} \tag{25}$$

This is the capacity of the AWGN channel with only a power constraint and no bandwidth constraint. It is important to see that the capacity is finite even though the bandwidth is not. The connection of this expression to energy efficient communication is explored in a homework exercise.

Looking Ahead

Starting next lecture, we shift gears and start looking at communication over wires. We start by looking at how a wireline channel affects voltage waveforms passing through it. We will be able to arrive at a discrete time wireline channel model by combining the effect of the wire on the voltage waveforms passing through it along with the DAC and ADC operations.