ECE 361: Digital Communications

Lecture 20: Sequential Communication over a Slow Fading Wireless Channel

Introduction

Consider the simple *slow* fading wireless channel we arrived at in the previous lecture:

$$y[m] = hx[m] + w[m], \quad m = 1, \ldots, N. \tag{1}$$

Here NT is the time scale (the product of N, the number of time samples communicated over, and T, the sampling period) of communication involved. The channel coefficient h is well modeled as independent of m if NT is much smaller than the coherence time T_c of the channel. This is a very common occurrence in many practical wireless communication systems and we begin our study of reliable communication over the wireless channel. We will start with sequential communication, much as we started out with the AWGN channel (cf. Lecture 3). The goal of this lecture is to be able to compare and contrast the simple slow fading channel in Equation (1) with the familiar AWGN channel. In particular, we will calculate the unreliability of communicating a single bit as a function of the SNR. The main conclusion is the observation of how poorly the unreliability decays with increasing SNR. This is especially stark when compared to the performance over the AWGN channel.

Comparison with AWGN Channel Model

The channel model in Equation (1) is quite similar to that of the AWGN channel model. But there are differences as well, with some aspects being more important than others.

1. The transmit and receive symbols are complex numbers (pair of voltages) as opposed to the real numbers in the AWGN channel. This is a relatively minor point and poses hardly any trouble to our calculations and analysis (as seen further in this lecture).

2. The channel "quality" h can be learnt by the receiver (through the transmission of pilot symbols, cf. Lecture 9), but is not known a priori to the communication engineer. This is a very important difference, as we will see further in this lecture. The important point is that the calculation of the unreliability level is now to be calculated in terms of the knowledge the communication engineer has about h: the *statistical* characterization of h.

In this lecture we make the following suppositions about the channel quality h:

1. the channel quality h is learnt very reliably at the receiver. Since we have several, namely N samples to communicate over, we could spend the first few samples in transmitting knows voltages (pilots) thus allowing the receiver to have a very good estimate of h. We will suppose that the receiver knows h exactly, and not bother to model the fact that there will be some error in the estimate of the channel as opposed to the true value.

2. a statistical characterization of h is available to the communication engineer: we study the relation between unreliability level and SNR in the context of a simple statistical model: *Rayleigh* fading.

Sequential Communication

Suppose we transmit a single information bit every time instant, using the modulation symbols $\pm\sqrt{E}$. We decode each bit separately over time as well. Focusing on a specific time m, we can write the received complex symbol as

$$\Re\left[y[m]\right] = \Re\left[h\right]x[m] + \Re\left[w[m]\right], \quad (2)$$
$$\Im\left[y[m]\right] = \Im\left[h\right]x[m] + \Im\left[w[m]\right]. \quad (3)$$

This follows since $x[m]$ is real (and restricted to be $\pm\sqrt{E}$). Since the receiver known $\Re\left[h\right]$ and $\Im\left[h\right]$, a sufficient statistic of the transmit symbol $x[m]$ is (cf. Lecture 5):

$$\begin{aligned}\tilde{y} &\stackrel{\text{def}}{=} \Re\left[h\right]\Re\left[y[m]\right] + \Im\left[h\right]\Im\left[y[m]\right] && (4)\\ &= \Re\left[h^* y[m]\right] && (5)\\ &= |h|^2 x[m] + \tilde{w}. && (6)\end{aligned}$$

Here we have denoted h^* as the *complex conjugate* of h: the real part of h and h^* are identical, but the imaginary part of h^* is the negative of that of h. Further,

$$\tilde{w} \stackrel{\text{def}}{=} \Re\left[h\right]\Re\left[w[m]\right] + \Im\left[h\right]\Im\left[w[m]\right] \quad (7)$$

is real valued and has Gaussian statistics: zero mean and variance $\frac{1}{2}|h|^2\sigma^2$. Now the ML receiver to detect $x[m]$ from $\tilde{y}$ is very clear: Equation (6) shows that the relation is simply that of an AWGN channel and we arrive at the nearest neighbor rule:

$$\text{decide } x[m] = +\sqrt{E} \text{ if } \tilde{y} > 0, \quad (8)$$

and vice versa if $\tilde{y} \leq 0$. The corresponding error probability is

$$Q\left(\sqrt{2|h|^2\mathsf{SNR}}\right), \quad (9)$$

where we have written $\mathsf{SNR} = E/\sigma^2$ as usual.

Average Error Probability

At this point, it is important to observe the nature of the unreliability level calculated in Equation (9) depends on the actual value of the channel quality (h) experienced during communication. But the communication engineer who has to decide how to set the operating value of SNR does not have access to the actual value of h. Only a statistical characterization of h is available to the engineer. One natural way to use the *dynamic* unreliability level calculated in Equation (9) to calculate a quantity useful to the communication engineer is to take the statistical average of the dynamic unreliability levels in Equation (9):[1]

$$P_e \stackrel{\text{def}}{=} \mathbb{E}\left[Q\left(\sqrt{2|h|^2\mathsf{SNR}}\right)\right], \tag{10}$$

where the average is with respect to the statistical characterization of h available to the communication engineer.

The average of the dynamic unreliability level in Equation (10) can be calculated based on the statistics of $|h|^2$. When h has the Rayleigh statistics we mean that the real and imaginary parts of h are i.i.d. Gaussian (zero mean and variance $\frac{A}{2}$ each). Here the quantity A stands for the *attenuation* (ratio of received energy to transmit energy in the signal) caused by the channel. It turns out that the statistics of $|h|^2$ are very simple – the density turns out to be exponential (most communication books will give you this result – just look for Rayleigh distribution in the index):

$$f_{|h|^2}(a) = \frac{1}{A} e^{-\frac{a}{A}}, \quad a \geq 0. \tag{11}$$

We can use the exponential statistics of $|h|^2$ to explicitly evaluate the average probability over the Rayleigh slow fading wireless channel:

$$\begin{aligned} P_e &= \int_0^\infty Q\left(\sqrt{2a\mathsf{SNR}}\right) \frac{1}{A} e^{-\frac{a}{A}}\, \mathrm{d}a & (12) \\ &= \frac{1}{2}\left(1 - \sqrt{\frac{A\mathsf{SNR}}{1 + A\mathsf{SNR}}}\right). & (13) \end{aligned}$$

The expression in Equation (13) is evaluated using the formula for "integrating-by-parts"; you can also take look at Equation 3.19 of the book *Fundamentals of Wireless Communication* by Tse and Viswanath. It is interesting to note that while the dynamic unreliability level did not have a concrete closed form expression (we could only write it as a Q function), the average of the dynamic unreliability level indeed has a simple formula.

[1] Note that this is not the only way to connect the dynamic unreliability level to one that depends only on the statistical characterization of the channel quality. Another approach would be to consider the smallest unreliability level that at least a fixed fraction, say 90%, of the channel qualities meet.

Average Unreliability vs SNR

Now we, as communication engineers, are in a position to decide what value of SNR to operate at given the need for a certain reliability level of communication. To do this, it will help to simplify the expression in Equation (13): using the Taylor series expansion

$$\sqrt{1+x} \approx 1 + \frac{x}{2}, \quad x \approx 0, \tag{14}$$

we can approximate the expression in Equation (13) as

$$P_e \approx \frac{1}{4A\mathsf{SNR}}, \quad A\mathsf{SNR} \gg 1. \tag{15}$$

Now we see the benefits of increasing SNR: for every doubling of SNR, the unreliability level only *halves*. This is in stark contrast to the behavior of the AWGN channel where the unreliability level *squared* for every doubling of SNR (cf. Lecture 3). To get a feel for how bad things are, let us consider a numerical example. If we desire a bit error probability of no more than 10^{-10} and have an attenuation of $A = 0.01$, then we are looking at a required SNR of $25 \cdot 10^{10}$! This corresponds to astronomical transmit powers – clearly physically impossible to meet.

Looking Ahead

We have seen that the performance of sequential communication is entirely unacceptable at physically attainable SNR levels. This is serious motivation to look for better strategies. One possibility is, continuing along the line of thought early in this course, to study block communication schemes. Block communication schemes improved the reliability level while maintaining non-zero communication rates. The goal of block communication was primarily to better exploit the statistical nature of the additive noise by mapping information packet directly to the transmit symbol vector (of high dimension). As such, it was a way to deal with additive noise. In the wireless channel, we have an additional source of noise: the channel quality h itself is random and it shows up in a *multiplicative* fashion. So, it is not entirely clear if block communication which was aimed at ameliorating the effects of additive noise would work too well in dealing with multiplicative noise. We get a better feel for this aspect in the next lecture where we see that block communication cannot improve the performance significantly beyond the expression in Equation (15).