ECE 361: Digital Communications

Lecture 21: Typical Error Event in a Slow Fading Wireless Channel

Introduction

We studied sequential communication over a slowly varying wireless channel in the previous lecture. The key feature of our channel was that it so slowly changing in time that it is practically time-invariant over the course of communication. Mathematically, our model is:

$$y[m] = hx[m] + w[m], \quad m = 1, \ldots, N. \tag{1}$$

We defined the probability of error with sequential communication over this channel as the average of the dynamic probability of error. With Rayleigh fading (i.e., $|h|^2$ has exponential density with mean A) we had the following relationship between SNR and unreliability of a single bit:

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{A\mathsf{SNR}}{1 + A\mathsf{SNR}}}\right) \tag{2}$$

$$\approx \frac{1}{4A\mathsf{SNR}}, \quad A\mathsf{SNR} \gg 1. \tag{3}$$

The main conclusion is that the probability of error decays only linearly with increase in SNR. This is in stark contrast to the situation in the AWGN channel (or a wireline channel) where we had an exponential decay. This, of course, requires huge improvements in the P_e vs SNR interdependence, for any communication over the wireless channel to be acceptable. As is clear from the Equation (3), there are two factors that cause errors over the wireless channel: the first is the additive noise w[m] (just as it was for the wireline channel) and the second (novel) factor h is multiplicative in nature. In this lecture we focus on isolating the effects of each of these factors in an attempt to figure out which one is more crucial. This insight will be used in later lectures to improve the poor relation between unreliability level and SNR that is exhibitited in Equation (3).

Sequential Communication

Consider sequential communication of a single bit x from the set $\pm\sqrt{E}$. We use the nearest neighbor rule on a real sufficient statistic

$$\tilde{y}[m] = \Re\left[\frac{h^*}{|h|}y[m]\right] \tag{4}$$

$$= |h|x[m] + \tilde{w}[m]. \tag{5}$$

Here $\tilde{w}[m]$ is real and Gaussian, with zero mean and variance $\frac{\sigma^2}{2}$. For example, the nearest neighbor rule at a time m (index is suppressed here) could be

$$\text{decide } x = +\sqrt{E} \text{ if } \tilde{y} > 0. \tag{6}$$

Now suppose we send $x = -\sqrt{E}$. Then an error occurs exactly when

$$\tilde{w} > |h|\sqrt{E}. \tag{7}$$

Observe that the error occurs due to the combined effect of $|h|$ and $\tilde{w}$. Thus there are two factors at play here when an error occurs. At a rough level we see that the smaller the $|h|$, the more likely the chance of an error. Let us consider only such events when $|h|$ is smaller than a threshold, denoted below by a_{th}:

$$|h|^2 < \frac{\sigma^2}{E} a_{\text{th}}. \tag{8}$$

One guess would be that such events, denoted henceforth as *outage*, would contribute significantly to errors (for appropriately chosen value of a_{th}).

A Suboptimal Receiver

Now consider a receiver that operates in a simple manner thus:

- in the event of an outage (i.e., the event in Equation (8) is satisfied), the receiver just gives up and refuses to make a decision based on the aposteriori probabilities. Thus the unreliability level of decoding the single bit correctly (based on the apriori probabilities) is 0.5.
- when not in outage (i.e., the value of $|h|$ is larger than the desired threshold), the receiver is the regular ML receiver. In this case, the unreliability level $Q\left(\sqrt{2|h|^2 \mathsf{SNR}}\right)$ (cf. Equation (10) in Lecture 19) is at least as small as $Q\left(\sqrt{2a_{\text{th}}}\right)$. If a_{th} is chosen larger enough (say, 18) then the unreliability level is very small (say, 10^{-10}).

This simple receiver based on a modification of the optimal ML receiver is suboptimal in general. The exact probability of making an error with this receiver is:

$$\begin{aligned} P_e &= \mathbb{P}\left[\text{Outage}\right] \cdot 0.5 + \mathbb{P}\left[\text{NoOutage}\right] \cdot Q\left(\sqrt{2a_{\text{th}}}\right) && (9) \\ &\approx \mathbb{P}\left[\text{Outage}\right] \cdot 0.5. && (10) \end{aligned}$$

This is because we have supposed that the threshold a_{th} is large enough that errors when outage does not occur are very unlikely. How likely is the outage event? Of course, this would depend on the specific statistics of $|h|^2$. For the Rayleigh fading model, the statistics of $|h|^2$ is exponential (with mean A) and

$$\mathbb{P}\left[|h|^2 < \frac{a_{\text{th}}}{\mathsf{SNR}}\right] = \int_0^{\frac{a_{\text{th}}}{\mathsf{SNR}}} \frac{1}{A} e^{-\frac{a}{A}}\, \mathrm{d}a \tag{11}$$

$$= 1 - e^{-\frac{a_{\text{th}}}{A\mathsf{SNR}}} \tag{12}$$

$$\approx \frac{a_{\text{th}}}{A\mathsf{SNR}}. \tag{13}$$

With $a_{\text{th}} = 18$ we see that the probability of outage is approximately

$$\mathbb{P}\left[\text{Outage}\right] \approx \frac{18}{A\mathsf{SNR}}. \tag{14}$$

Based on the calculations so far, a few observations are in order now.

- The error probability of the receiver with the simple modification based on the outage event is dominated by the probability of the outage event. This is more or less by design, since we defined the outage event as one where errors are most likely.
- The probability of outage (cf. Equation (14)) while larger than the expression for the true error probability (cf. Equation (3)), has the *same* qualitative behavior with SNR: i.e., it decays inversely with SNR. This is a remarkable happenstance and allows us to conclude that the outage is the *typical* error event. Essentially we have seen that the fading channel magnitude being small is the dominant way errors occur.

We are now in a position to better evaluate our options to improve the behavior of unreliability level with SNR: we just need to reduce the chance of being in outage. Indeed, we will take this approach in the next several lectures. Before we get to this, it is worth taking a more fundamental look at our approach so far. We have made the observations regarding outage being the typical error event based on the assumption of sequential communication. We know from our earlier discussion (lectures 4 through 8) that block communication yields significantly better performance than that of sequential communication. It is worth redoing our outage analysis in the context of the fundamentally superior block coding context. This is the next topic.

Outage and Block Communication

In an earlier lecture we have derived the *capacity* of the (real) AWGN channel. What is the capacity of the *complex* based band AWGN channel:

$$y[m] = hx[m] + w[m], \tag{15}$$

i.e., the slow fading wireless channel if the complex channel gain h were known to the communication engineer who is designing the encoder and decoder rules? Here we have the usual power constraint (slightly modified to suit the complex transmit symbol):

$$\sum_{m=1}^{N} |x[m]|^2 \leq P. \tag{16}$$

There is a simple way to convert the complex base band AWGN channel into the more familiar (real) AWGN channel: consider the invertible transformation

$$\begin{aligned} \tilde{y}[m] &= \frac{h^*}{|h|} y[m] && (17) \\ &= |h| x[m] + \tilde{w}[m]. && (18) \end{aligned}$$

Here $\tilde{w}[m]$ has the same statistics as that of w (this is similar to the statement made in our discussion on OFDM): i.e., it is a complex random variable with both real and imaginary parts independent and Gaussian with zero mean and variance $\frac{\sigma^2}{2}$. The complex AWGN channel in Equation (18) can be expanded out as

$$\begin{aligned} \Re\left[\tilde{y}[m]\right] &= |h| \Re\left[x[m]\right] + \Re\left[\tilde{w}[m]\right], && (19) \\ \Im\left[\tilde{y}[m]\right] &= |h| \Im\left[x[m]\right] + \Im\left[\tilde{w}[m]\right]. && (20) \end{aligned}$$

This is now a regular (real) AWGN channel where we get to transmit and receive two symbols for every time instant. The power constraint in Equation (16) means that the power per symbol in this (real) AWGN channel is $P/2$. The capacity of such a channel is simply *twice* that possible over the (real) AWGN channel with additive noise variance $\sigma^2/2$ and the largest rate of reliable communication over such a channel would be:

$$C = \log_2 \left(1 + \frac{|h|^2 P}{\sigma^2}\right) \text{ bits/sample.} \tag{21}$$

Physically, one can understand the transmission of two symbols per time sample as representing the two signals transmitted in quadrature over the passband channel.

Here again, the capacity of the channel has the same interpretation as that for the (real) AWGN channel: i.e., arbitrarily reliable communication is possible at data rates below capacity and communication is arbitrarily unreliable at data rates above capacity. This is the situation in the complex AWGN case. But in the slow fading wireless channel, the communication engineer is not privy to the exact value of h and hence the corresponding *received* SNR of the channel. This is tantamount to fixing the data rate to R bits/sample ahead of time and using good AWGN encoding and decoding procedures. Depending on the dynamic channel quality we have one of two possible outcomes:

1. If $R < C$, then communication is arbitrarily reliable.
2. If $R > C$, then communication is arbitrarily unreliable.

The overall probability of error with block coding therefore is:

$$
\begin{aligned}
P_e &= \mathbb{P}\left[R > C\right] \cdot 1 + \mathbb{P}\left[R < C\right] \cdot 0 && (22) \\
&= \mathbb{P}\left[R > \log_2\left(1 + |h|^2 SNR\right)\right]. && (23)
\end{aligned}
$$

The goal of block communication was to avoid the deleterious effects of additive noise w and, indeed, we see that the error probability now depends only on the channel magnitude h. We can explicitly calculate the probability of error with rate efficient block communication for the Rayleigh fading model: In this case

$$
\begin{aligned}
P_e &= 1 - e^{-\frac{2^R - 1}{A\mathsf{SNR}}} && (24) \\
&\approx \frac{2^R - 1}{A\mathsf{SNR}}. && (25)
\end{aligned}
$$

Once again, a few observations are in order.

1. Even in the context of rate efficient block communication, the typical error event is the same as that defined earlier: outage. The only change is that the threshold a_{th} changed (from a somewhat arbitrary setting of 18 to $\left(2^R - 2\right)/\mathsf{SNR}$).
2. Comparing Equation (25) with Equation (14), we see that even with rate efficient block communication, the behavior of unreliability with SNR is the same as before: it decays linearly with SNR.

So now we can safely conclude that outage . Hence, the whole analysis done in this lecture leads to the final conclusion: the only way to improve error performance over the slow fading wireless channel is to somehow reduce the chance of an outage event!

Looking Ahead

We conclude that outage is a *fundamentally typical* error event. To improve the overall unreliability level (for a given SNR) the only way is to reduce the chance of outage (for the same SNR). We investigate several different ways to do this starting the next lecture.