

# ECE 361: Fundamentals of Digital Communications

## Lecture 22: Time Diversity

### Introduction

We have seen that the communication (even with coding) over a slow fading flat wireless channel has very poor reliability. This is because there is a significant probability that the channel is in outage and this event dominates the total error event (which also include the effect of additive noise). The key to improving the reliability of reception (and this is really required for wireless systems to work as desired in practice) is to reduce the chance that the channel is in outage. This is done by communicating over “different” channels and the name of the game is to harness the *diversity* available in the wireless channel. There are three main types: temporal, frequency and antennas. We will see each one over separate lectures, starting time diversity.

### Time Diversity Channel

The temporal diversity occurs in a wireless channel due to mobility. The idea is to code and communicate over the order of time over which the channel changes (called the *coherence* time). This means that we need enough delay tolerance by the data being communicated. A simple single-tap model that captures time diversity is the following:

$$y_\ell = h_\ell x_\ell + w_\ell, \quad \ell = 1, \dots, L. \quad (1)$$

Here the index  $\ell$  represents a single time sample over *different* coherence time intervals. By interleaving across different coherence time intervals, we can get to the time diversity channel in Equation (1). A good statistical model for the channel coefficients  $h_1, \dots, h_L$  is that they are all independent and identically distributed. Further more, in a environment with lots of multipath we can suppose that they are complex Gaussian random variables (the so-called Rayleigh fading).

### A Single Bit Over a Time Diversity Channel

For ease of notation, we start out with  $L = 2$  and just a single bit to transmit. The simplest way to do this is to *repeat* the same symbol: i.e., we set

$$x_1 = x_2 = x \pm \sqrt{E}. \quad (2)$$

At the receiver, we have *four* (real) voltages:

$$\Re [y_1] = \Re [h_1] x + \Re [w_1], \quad (3)$$

$$\Im [y_1] = \Im [h_1] x + \Im [w_1], \quad (4)$$

$$\Re [y_2] = \Re [h_2] x + \Re [w_2], \quad (5)$$

$$\Im [y_2] = \Im [h_2] x + \Im [w_2]. \quad (6)$$

As usual, we suppose *coherent* reception, i.e., the receiver has full knowledge of the exact channel coefficients  $h_1, h_2$ . By now, it is quite clear that the receiver might as well take the appropriate weighted linear combination to generate a single real voltage and then make the decision with respect to the single transmit voltage  $x$ . This is the matched filter operation:

$$y^{\text{MF}} = \Re [y_1] \Re [h_1] + \Im [y_1] \Im [h_1] + \Re [y_2] \Re [h_2] + \Im [y_2] \Im [h_2]. \quad (7)$$

Using the complex number notation,

$$y^{\text{MF}} = \Re [h_1^* y_1] + \Re [h_2^* y_2], \quad (8)$$

$$= (|h_1|^2 + |h_2|^2) x + \tilde{w}. \quad (9)$$

Here  $\tilde{w}$  is a real Gaussian random variable because it is the sum of four independent and identically distributed Gaussians, but they get scaled by the real and imaginary parts of  $h_1$  and  $h_2$ . Therefore  $\tilde{w}$  is zero mean and has variance:

$$\text{Var}(\tilde{w}) = \frac{\sigma^2}{2} (|h_1|^2 + |h_2|^2). \quad (10)$$

The decision rule is now simply the nearest neighbor rule:

$$\text{decide } x = +\sqrt{E} \text{ if } y^{\text{MF}} > 0, \quad (11)$$

and zero, otherwise.

## Probability of Error

The *dynamic* error probability is now readily calculated (since the channel – cf. Equation (9) – is now of an AWGN type):

$$P_e^{\text{dynamic}} = Q \left( \frac{(|h_1|^2 + |h_2|^2) \sqrt{E}}{\frac{\sigma}{\sqrt{2}} (\sqrt{|h_1|^2 + |h_2|^2})} \right). \quad (12)$$

From the transmitter's perspective, it makes sense to calculate the *average* error probability, averaged over the statistics of the two channels  $h_1, h_2$ :

$$P_e = \mathbb{E} \left[ Q \left( \sqrt{2\text{SNR}} (|h_1|^2 + |h_2|^2) \right) \right]. \quad (13)$$

For the statistical model we suggested earlier (independent Rayleigh fading), this expression can be evaluated exactly:

$$P_e = \left( \frac{1 - \mu}{2} \right)^2 (1 + \mu) \quad (14)$$

where

$$\mu \stackrel{\text{def}}{=} \sqrt{\frac{ASNR}{1 + ASNR}}. \quad (15)$$

Much like we did in earlier lectures, we can try to simplify Equation (14) in order to better see the relationship between unreliability level and  $SNR$ . Specifically, at high  $SNR$  (i.e.,  $SNR \gg 1$ ), we have

$$(1 - \mu) \approx \frac{1}{ASNR} \quad (16)$$

$$(1 + \mu) \approx 2. \quad (17)$$

Substituting these approximations into Equation (14), we conclude that

$$P_e = \frac{1}{8A^2SNR^2}. \quad (18)$$

## Diversity Gain

Equation (18) that captures the approximate relationship between transmit  $SNR$  and unreliability level is instructive: we see that a doubling in  $SNR$  reduces the unreliability by a *quarter* factor. This is significant improvement when compared to the earlier case of no diversity: in that case, a doubling of  $SNR$  only decreased  $P_e$  by a factor of half. The exponent of  $SNR$  in Equation (18) is referred to as ‘*diversity gain*’, and in this case it is equal to 2.

## Generalization

We have worked with  $L = 2$  for simplicity. For a general  $L$ , we have the natural extension to repetition coding at the transmitter and matched filter reception (with nearest neighbor decoding) at the receiver. Equation (9) generalizes to

$$y^{\text{MF}} = \left( \sum_{\ell=1}^L |h_\ell|^2 \right) x + \tilde{w}, \quad (19)$$

where  $\tilde{w}$  is real Gaussian with zero mean and variance (cf. Equation (10))

$$\left( \sum_{\ell=1}^L |h_\ell|^2 \right) \frac{\sigma^2}{2}. \quad (20)$$

Thus, Equation (13) generalizes to

$$P_e = \mathbb{E} \left[ Q \left( \sqrt{2SNR \left( \sum_{\ell=1}^L |h_\ell|^2 \right)} \right) \right]. \quad (21)$$

Again, there is an exact expression for the unreliability level when the channel coefficients are independent Rayleigh distributed (as a generalization of Equation (14)):

$$\left( \frac{1-\mu}{2} \right)^L \sum_{k=0}^{L-1} \binom{L-1+k}{k} \left( \frac{1+\mu}{2} \right)^k, \quad (22)$$

where  $\mu$  is as before (cf. Equation (15)). Finally, we can look for a high SNR approximation. Equation (18) now becomes

$$P_e \approx \binom{2L-1}{L} \frac{1}{(4ASN R)^L}. \quad (23)$$

The diversity gain is now  $L$ : doubling of SNR reduces the unreliability by a factor of

$$\frac{1}{2^L}. \quad (24)$$

Figure 1 depicts the power of having time diversity. It is instructive to see that to even have a reliably high unreliability level (per bit) of  $10^{-5}$  we need a SNR of almost 40dB with first order diversity, but this reduces to just 12 dB with fifth order diversity. The diversity gain is even starker at lower unreliability levels.

## Looking Ahead

In the next two lectures we will look at two other modes of diversity: frequency and antennas. These come in handy particularly when time diversity is either absent or cannot be effectively harnessed due to the tight delay constraint of the data (say, gaming applications).

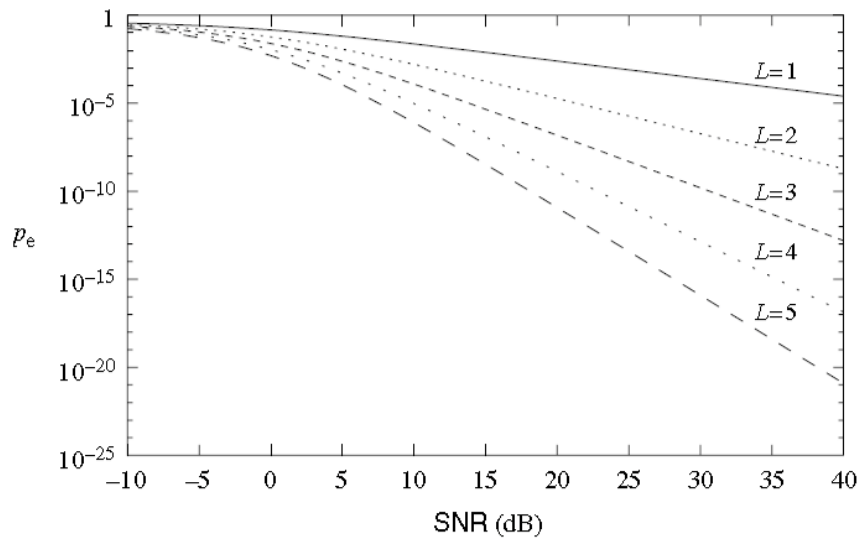


Figure 1: Bit error probability as a function of different diversity orders  $L$ .