ECE361 Fall 2001 **Midterm 1**

Instructions:

1. The midterm has total worth 50 points. Use your time judiciously; you have 1.5 hours for this exam. Good luck.
2. This is a closed notes, closed book exam. You also will not need any calculators. Please have only your pens, pencils, erasers with you.
3. Along with the midterm you have a midterm review sheet for the instructor and the TA. You can use this opportunity to provide feedback to the instructor and the TA. If there is an overwhelming vote towards or against a certain policy, then this vote will translate into an action. Do *not* write your name on this sheet and submit it into a pile separate from the midterms.

Questions.

1. (12 points). **True or False**.

$$
\begin{aligned}
\text{Scoring} &= +2 \text{ for correct answer with explanation} \\
&= +1 \text{ for correct answer with no (or incorrect) explanation} \\
&= -1 \text{ for incorrect answer} \\
&= 0 \text{ if left unanswered}
\end{aligned}
$$

(a) X and Y be two random variables. Then $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$ if X and Y are uncorrelated.

(b) Suppose X and Y are now Gaussian random variables with zero mean and unit variance. Suppose further that they are uncorrelated. Then X and Y are independent.

(c) Suppose X is some random variable and construct the random process $Y(t) \stackrel{\text{def}}{=} Xt$ for all t. Then this process is *strict sense stationary*.

(d) Suppose E_1 and E_2 are two events of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If E_1 and E_2 are disjoint, then they are independent.

(e) Suppose two events E_3 and E_4 are independent. Then it must be that

$$\mathbb{P}[E_3 \cup E_4] \leq \mathbb{P}[E_3]\,\mathbb{P}[E_4] \ .$$

(f) Suppose X and Y are independent zero mean, Gaussian random variables with unit variance. Then $X+Y$ and $X-Y$ are independent.

2. (Total points 11).
In class we saw two types of modulation of pulses: amplitude and position. Consider the following modulation scheme (which is somewhat of a hybrid) for symbols that are n bits long. The first k bits picks the amplitude A_i and the last $n-k$ bits the phase ϕ_j. The modulated waveform is

$$A_i \cos\left(\frac{2\pi t}{T} + \phi_j\right)$$

on $[0, T]$. Since n bits are directly modulated, this is a block modulation scheme and the number of messages is 2^n.

   (a) (6 points) Find an orthonormal basis for the waveforms. You should try to find the smallest number of orthonormal waveforms that forms a basis.
   (b) (2 points) What is the dimension of the space spanned by the waveforms?
   (c) (3 points) What is the geometry of the signal constellation? By this, I mean draw/describe the constellation diagram. (*Hint: This has an elegant answer for all* n).

3. (Total points 12).
Consider a communication scheme where one of 4 messages has to be communicated. The signals corresponding to these 4 messages are chosen to be the following. They are all zero for time $t < 0$ and time $t > T$. For $t \in [0, T]$ they are given below.

$$\begin{aligned} s_1(t) &= \sin\left(\frac{2\pi t}{T}\right) \\ s_2(t) &= \cos\left(\frac{2\pi t}{T}\right) \\ s_3(t) &= \begin{cases} 1 & t \in \left[0, \frac{T}{2}\right] \\ 0 & t \in \left(\frac{T}{2}, T\right] \end{cases} \\ s_4(t) &= \begin{cases} -1 & t \in \left[0, \frac{T}{2}\right] \\ 0 & t \in \left(\frac{T}{2}, T\right] \end{cases} \end{aligned}$$

   (a) (7 points). Find an orthonormal basis for the signal waveforms and hence design a set of decorrelators to extract a set of sufficient statistics for optimal detection. How many sufficient statistics are needed?
   (b) (5 points). Now suppose the received signal is projected onto the signal waveforms $s_1(t), \ldots, s_4(t)$ directly. Do these four components provide sufficient information for optimal detection? Why?

4. (Total points 15).
A transmitted symbol X is equally likely to be $+1$ or -1. and is received at n antennas:

$$Y_i = X + Z_i, \; i = 1, \ldots, n \, ,$$

where the additive noise Z_i at antenna i is Gaussian with mean zero and variance σ^2 and is independent across all antennas and of the transmitted signal X.

(a) (2 points). Focus first on one of the antennas (say, i) in isolation. Find the optimal receiver (this is the receiver that minimizes the average probability of error, as we defined in class) that decides on the transmitted symbol based solely on the signal received at antenna i (namely Y_i). Let us denote the output of this receiver as $\hat{X}_i$, an estimate of the transmitted signal X. This estimate takes values in $\{+1, -1\}$ based on what the received signal Y_i is. Find the resulting probability of error (the error event occurs when $\hat{X}_i \neq X$).

(b) (7 points). Now we would like to combine the single antenna focused estimates $\hat{X}_1, \ldots, \hat{X}_n$ to come up with an overall estimate $\hat{X}$ of the transmitted signal. Find the optimal way of combining $\hat{X}_1, \ldots, \hat{X}_n$ that minimizes the probability of detecting X wrongly. Give an expression for the probability of error.

(c) (6 points). Is the estimate $\hat{X}$ you got in the previous part the same as the optimal receiver output that bases its decision on the *received signals* $Y_1, \ldots, Y_n$? (again, optimal in the sense of minimizing probability of error). If so, explain. If not, compute the optimal decision rule and the resulting probability of error.