# ECE361 Fall 2002 — Midterm 1

Instructions:

1. The midterm has total worth 40 points. Use your time judiciously; you have 1 hour and 20 minutes for this exam. Good luck.
2. This is a closed notes, closed book exam. You also will not need any calculators. Please have only your pens, pencils, erasers with you.

1. (13 points)

   (a) (2 points) Answer *True* or *False* with an explanation. No points for correct answers with incorrect explanation.
   Suppose $X$ and $Y$ are Gaussian random variables with zero mean and unit variance. Suppose further that they are uncorrelated. Then $X$ and $Y$ are independent.

   (b) (2 points) Answer *True* or *False* with an explanation. No points for correct answers with incorrect explanation.
   Suppose $E_1$ and $E_2$ are two events of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If $E_1$ and $E_2$ are disjoint, then they are independent.

   (c) (2 points) Answer *True* or *False* with an explanation. No points for correct answers with incorrect explanation.
   Suppose two events $E_3$ and $E_4$ are independent. Then it must be that

   $$\mathbb{P}\left[E_3 \cup E_4\right] \leq \mathbb{P}\left[E_3\right] \mathbb{P}\left[E_4\right] \ .$$

   (d) (3 points) Consider two signals in a 2-dimensional space represented in polar coordinates as follows:

   $$\mathbf{s}_1 = (r_1, \theta_1) \text{ and } \mathbf{s}_2 = (r_2, \theta_2) \, .$$

   If these two signals are used equally likely in an AWGN channel with variance $\frac{\mathcal{N}_0}{2}$, what is the ML receiver and the corresponding error probability?

   (e) (4 points) Suppose $X$ and $Y$ are independent Gaussian zero mean random variables. Find the joint distribution of the two random variables $(R, \theta)$ defined as follows:

   $$R = \sqrt{X^2 + Y^2} \text{ and } \theta = \tan^{-1}\left(\frac{X}{Y}\right) .$$

2. (10 points) In ECE 359, you have read about phase and amplitude modulation. One way to represent both of these in the language we have been talking in this course is the following. Consider the following mapping of $M = 2^n$ messages to the signal waveforms as follows. Each of the $m$ messages can be represented by a vector of bits

of length $n$. (There are $2^n$ possible such bit vectors corresponding to the number of messages). The bit vector $\mathbf{b} = (b_1, \dots, b_n)$ gets mapped to the waveform:

$$A_{\mathbf{b}} \cos\left(\frac{2\pi t}{T} + \phi_{\mathbf{b}}\right) ,$$

on $[0, T]$. Here the amplitude of the signal $A$ depends on the message (represented by the bit vector $\mathbf{b}$), as does the phase $\phi$. If $A$ did not depend on $\mathbf{b}$ then we have pure phase modulation and if $\phi$ did not depend on $\mathbf{b}$ then we have pure amplitude modulation.

(a) (5 points) Find an orthonormal basis for the waveforms. You should try to find the smallest number of orthonormal waveforms that forms a basis.

(b) (2 points) What is the dimension of the space spanned by the waveforms?

(c) (3 points) What is the geometry of the signal constellation? By this, I mean draw/describe the constellation diagram. (*Hint: This has an elegant answer for all $n$*).

3. (5 points) Consider a communication scheme where one of 4 messages has to be communicated. The signals corresponding to these 4 messages are chosen to be the following. They are all zero for time $t < 0$ and time $t > T$. For $t \in [0, T]$ they are given below.

$$\begin{aligned} s_1(t) &= \sin\left(\frac{2\pi t}{T}\right) \\ s_2(t) &= \sin\left(\frac{2\pi t}{T} + \frac{\pi}{4}\right) \\ s_3(t) &= \begin{cases} 1 & t \in \left[0, \frac{T}{2}\right] \\ 0 & t \in \left(\frac{T}{2}, T\right] \end{cases} \\ s_4(t) &= \cos\left(\frac{2\pi t}{T}\right) \end{aligned}$$

Find an orthonormal basis for the signal waveforms. What is the minimum number of basis signals you need?

4. (7 points) Consider a communication system with two equally likely messages $m_0$ and $m_1$ to be conveyed. Suppose the message $m_0$ is associated with a voltage level $+A$ and the message $m_1$ is associated with a voltage level $-A$. The transmitted voltage (either $+A$ or $-A$) is denoted by the random variable $s$. Consider the following two separate scenarios of how this transmitted voltage is received. Answer each question separately.

(a) (2 points). Two voltages are measured at the receiver and the model of the received voltages (denoted by $y_1$ and $y_2$) is as follows:

$$\begin{aligned} y_1 &= s + n_1 , \\ y_2 &= n_1 . \end{aligned}$$

Here $n_1$ is a zero Gaussian random variable with unit variance that is independent of the transmitted voltage $s$. Derive the structure of the optimal receiver *explicitly* (i.e., do not simply leave it as conditional probabilities) and the corresponding probability of detection error.

(b) (5 points) The model for the two voltages received are as follows:

$$\begin{aligned} y_1 &= s + n_1 - n_2 \ , \\ y_2 &= n_1 + n_2 \ . \end{aligned}$$

Here $n_1$ and $n_2$ are independent zero mean Gaussian random variables with unit variance that are jointly independent of the transmitted voltage $s$. Derive the structure of the optimal receiver and compute the corresponding probability of detection error.

5. (5 points) Let us reconsider Problem. 4 of your sample midterm (also HW4) from last week. A transmitted symbol $X$ is equally likely to be $+1$ or $-1$. and is received at $n$ antennas:

$$Y_i = X + Z_i, \ i = 1, \ldots, n \ ,$$

where the additive noise $Z_i$ at antenna $i$ is Gaussian with mean zero and variance $\sigma^2$ and is independent across all antennas and of the transmitted signal $X$. In HW4, you found the optimal receiver using each antenna in isolation. In this problem, you are asked to find the optimal (ML) receiver that uses all the antennas *jointly*. Also compute the resulting probability of error. *Hint*: Write the overall channel as a *vector* channel and observe that there are only two messages in this communication system.