September 20, 2006

Digital Modulation

- After possible source and error control encoding, we have a sequence $\{m_n\}$ of message symbols to be transmitted on the channel. The message symbols are assumed to come from a finite alphabet, say $\{0, 1, \dots, M-1\}$. In the simplest case of binary signaling, $M = 2$. Each symbol in the sequence is assigned to one of M waveforms $\{s_0(t), \dots, s_{M-1}(t)\}$.

- *Memoryless modulation versus modulation with memory.* If the symbol to waveform mapping is fixed from one interval to the next, i.e., $m \mapsto s_m(t)$, then the modulation is memoryless. If the mapping from symbol to waveform in the n-th symbol interval depends on previously transmitted symbols (or waveforms) then the modulation is said to have memory.

- For memoryless modulation, to send the sequence $\{m_n\}$ of symbols at the rate of $1/T_s$ symbols per second, we transmit the signal

$$s(t) = \sum_n s_{m_n}(t - nT_s) \, . \tag{1}$$

- *Linear versus nonlinear modulation.* A digital modulation scheme is said to be linear if we can write the mapping from the sequence of symbols $\{m_n\}$ to the transmitted signal $s(t)$ as concatenation of a mapping from the sequence $\{m_n\}$ to a complex sequence $\{c_n\}$, followed by a *linear* mapping from $\{c_n\}$ to $s(t)$. Otherwise the modulation is nonlinear.

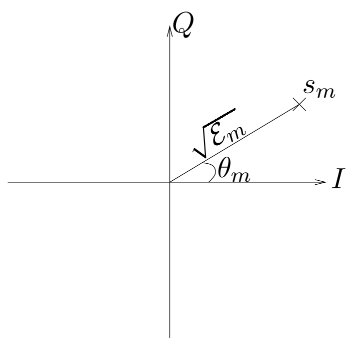
Linear Memoryless Modulation

- In this case, the mapping from symbols to waveforms can be written in complex baseband as:

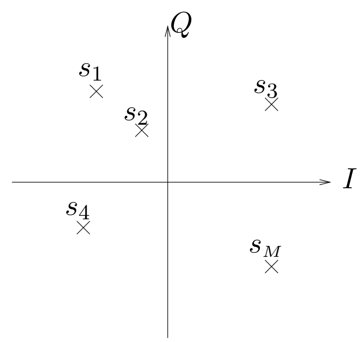
$$s_m(t) = \sqrt{\mathcal{E}_m}\, e^{j\theta_m} g(t) \, , \quad m = 1, 2, \dots, M \, , \tag{2}$$

where $g(t)$ is a real-valued, unit energy, pulse shaping waveform.

- The signal $s_m(t)$ can be represented by a point in the complex plane, i.e., the signal space corresponding to a symbol interval is a 1-d (complex) space with basis function $g(t)$.



Representation of $s_m(t)$



Signal constellation

- In real passband,

$$\tilde{s}_m(t) = \mathsf{Re}[\sqrt{2}s_m(t)\, e^{j2\pi f_c t}] = \sqrt{2\mathcal{E}_m}\ \cos(2\pi f_c t + \theta_m) g(t)\ . \tag{3}$$

- As we saw in class, the signal energy is the same in both the real passband and complex baseband domains and equals $\mathcal{E}_m$.

- The average symbol energy for the constellation is given by

$$\mathcal{E}_s = \frac{1}{M} \sum_{m=1}^{M} \mathcal{E}_m\ . \tag{4}$$

- The average bit energy for the constellation (assuming that $M = 2^\nu$, for some integer ν) is given by

$$\mathcal{E}_\mathrm{b} = \frac{\mathcal{E}_\mathrm{s}}{\log_2 M} = \frac{\mathcal{E}_\mathrm{s}}{\nu}\ . \tag{5}$$

- The distance between signals s_k and s_m is $d_{k,m} = \|s_k - s_m\|$, and the minimum distance is given by

$$d_\mathrm{min} = \min_{k,m} d_{k,m}\ . \tag{6}$$

- A measure of goodness of the constellation is the ratio

$$\zeta = \frac{d^2_\mathrm{min}}{\mathcal{E}_\mathrm{b}}\ . \tag{7}$$

  Note that ζ is independent of scaling of the constellation.

- Some commonly used signal constellations are:

  - *Pulse Amplitude Modulation (PAM).* Information only in amplitude:

$$\theta_m = 0 \text{ and } \sqrt{\mathcal{E}_m} = (2m - 1 - M)\frac{d}{2}, \quad m = 1, 2, \ldots, M\ . \tag{8}$$

    We can compute ζ as a function of M. For example, $\zeta = 4$ for $M = 2$.

  - *Phase Modulation or Phase Shift Keying (PSK).* Information only in phase:

$$\theta_m = \frac{2\pi m}{M} \text{ and } \mathcal{E}_m = \mathcal{E}\ , \quad m = 1, 2, \ldots, M\ . \tag{9}$$

    We showed in class that

$$d_\mathrm{min} = \sqrt{2\mathcal{E}\left(1 - \cos\tfrac{2\pi}{M}\right)} \quad \Longrightarrow \quad \zeta = 2\log_2 M\left(1 - \cos\tfrac{2\pi}{M}\right)$$

    For QPSK, $\zeta = 4$ (as in BPSK).

  - *Quadrature Amplitude Modulation (QAM).* Information in phase and amplitude. We can design constellations to maximize ζ for a given M. Rectangular constellations are convenient for demodulation. For rectangular 16-QAM, $\zeta = 1.6$.

Orthogonal Memoryless Modulation

- Here the signal set is given by

$$s_m(t) = \sqrt{\mathcal{E}}\, g_m(t), \quad m = 1, 2, \ldots, M \tag{10}$$

where $\{g_m(t)\}$ are (possibly complex) unit energy signals, i.e., $\|g_m(t)\| = 1$.

- The correlation between signals $s_k(t)$ and $s_m(t)$ is given by:

$$\rho_{km} = \frac{\langle s_k(t), s_m(t)\rangle}{\mathcal{E}} = \langle g_k(t), g_m(t)\rangle \tag{11}$$

- There are two kinds of orthogonality:
  - ∘ Orthogonality only in the real component of the correlation, i.e. $\mathsf{Re}\{\rho_{km}\} = 0$, for $k \neq m$. This form of orthogonality is enough for coherent demodulation.
  - ∘ Complete orthogonality, i.e., $\rho_{km} = 0$, for $k \neq m$. This is required for noncoherent demodulation.

- *Examples of orthogonal signal sets*
  - ∘ *Separation in time:*

$$g_m(t) = g\,(t - (m-1)T_\mathrm{s}/M) \tag{12}$$

where $g(t)$ is such that $\langle g(t - kT_\mathrm{s}/M), g(t - mT_\mathrm{s}/M)\rangle = \delta_{km}$. For example, $g(t) = p_{T_\mathrm{s}/M}(t)$, a rectangular pulse of width T_s/M.

This signal set is completely orthogonal. We can also create a signal set of twice the size which satisfies orthogonality only in the real component of the correlation by adding $\{jg_m(t)\}$ to the above signal set as we saw in class.

  - ∘ *Separation in frequency:*

$$g_m(t) = e^{j2\pi(m-1)\Delta_f t}\; p_{T_\mathrm{s}}(t) \tag{13}$$

It is easy to show that

$$\rho_{km} = \mathrm{sinc}[T_\mathrm{s}(k-m)\Delta_f]\; e^{j\pi T_\mathrm{s}(k-m)\Delta_f} \tag{14}$$

and that

$$\mathsf{Re}\{\rho_{km}\} = \mathrm{sinc}[2T_\mathrm{s}(k-m)\Delta_f]\,. \tag{15}$$

Thus the smallest value of Δ_f such that $\rho_{km} = 0$, for $k \neq m$, is $1/T_\mathrm{s}$, and such that $\mathsf{Re}\{\rho_{km}\} = 0$, for $k \neq m$, is $1/2T_\mathrm{s}$.

  - ∘ *Separation in time and frequency:* One way to do this is to pick $\{g_m(t)\}$ to be the Walsh-Hadamard functions on $[0, T_s]$ as we saw in class.