University of Illinois | **First Mid-Semester Exam** | **ECE 361**
of Illinois | **Page 1 of 3** | **Spring 2001**

Thursday, February 22, 2001
1:00 p.m. — 2:20 p.m.
161 Everitt Laboratory

Instructions:

There are **four problems** on this examination worth a total of 125 points.

One page of notes allowed. No other notes, books, tables of integrals, and calculators/personal computers permitted.

Show all your work in the exam booklet provided. Answers without appropriate justification will receive no credit.

Notation and (possibly) useful formulas

$\Phi(x)$ = cumulative probability distribution function for zero-mean unit-variance (i.e., standard) Gaussian random variable

$Q(x) = 1 - \Phi(x)$

$P_{e,i}$ = probability of error given that signal $s_i(t)$ was transmitted

AWGN denotes additive white Gaussian noise with (two-sided) power spectral density $N_0/2$. This random process is independent of the choice of transmitted signal. The mean of this process is 0.

$\text{rect}(t) =$		$\begin{cases} 1 & \text{if } -1/2 \le t \le 1/2, \\ 0, & \text{otherwise.} \end{cases}$	$p_T(t) =$		$\begin{cases} 1, & \text{if } 0 \le t < T, \\ 0, & \text{otherwise.} \end{cases}$
$X(t)$	$\leftrightarrow$	$x(-f)$	$x(\alpha t)$	$\leftrightarrow$	$\lvert\alpha\rvert^{-1}\cdot X(\alpha^{-1}f)$
$x^*(t)$	$\leftrightarrow$	$X^*(-f)$	$x^*(-t)$	$\leftrightarrow$	$X^*(f)$
$x(t-t_0)$	$\leftrightarrow$	$X(f)\cdot\exp(-j2\pi f t_0)$	$x(t)\cdot\exp(j2\pi f_0 t)$	$\leftrightarrow$	$X(f-f_0)$
$x(t)\cdot\cos(2\pi f_0 t)$	$\leftrightarrow$	$(^1/_2)[X(f+f_0) + X(f-f_0)]$	$x(t)\cdot\sin(2\pi f_0 t)$	$\leftrightarrow$	$(^j/_2)[X(f+f_0) - X(f-f_0)]$
$\frac{d}{dt}\, x(t)$	$\leftrightarrow$	$(j2\pi f)\cdot X(f)$	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\leftrightarrow$	$(j2\pi f)^{-1}\cdot X(f)+X(0)\cdot\delta(f)/2$

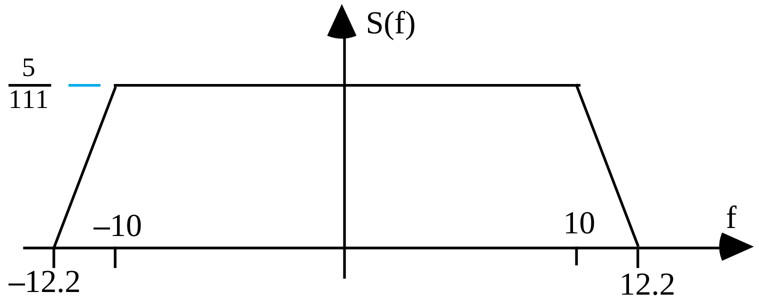
x∗y = \I(−∞,∞,x(τ)•y(t−τ)dτ) ↔ X(f)•Y(f) x(t)•y(t) ↔

$\int_{-\infty}^{\infty} X(\lambda)\cdot Y(f-\lambda)d\lambda$

$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$=$	$\int_{-\infty}^{\infty} \lvert X(f)\rvert^2 df$	$\int_{-\infty}^{\infty} x(t)y^*(t)dt$	$=$	$\int_{-\infty}^{\infty} X(f)Y^*(f)df$
$R_{x,y}(\tau)$	$=$	$\int_{-\infty}^{\infty} x(t+\tau)\cdot y^*(t)\, dt$	$R_{x,y}(\tau)$	$\leftrightarrow$	$S_{x,y}(f) = X(f)Y^*(f)$
$\text{rect}(t/T)$	$\leftrightarrow$	$T\cdot\text{sinc}(fT)$	$W\cdot\text{sinc}(Wt)$	$\leftrightarrow$	$\text{rect}(f/W)$

where $\text{sinc}(z) = \frac{\sin \pi z}{\pi z}$ is the *sinc function*

1. (20 points) A signal s(t) has Fourier transform S(f) as sketched below. Find s(t). In order to receive full credit, you must specify s(t) for all t, $-\infty < t < \infty$.



2. (20 points) The input to a linear time-invariant system with impulse response

$$h(t) = \begin{cases} \exp(-t), & 0 \le t < \infty, \\ 0, & t < 0, \end{cases}$$

is a zero-mean WSS Gaussian random process $\{\mathbf{X}(t) : -\infty < t < \infty\}$ with autocorrelation function

$$R_{\mathbf{X}}(\tau) = \exp(-2|\tau|), -\infty < \tau < \infty.$$

Let $\{\mathbf{Y}(t) : -\infty < t < \infty\}$ denote the output process. What is the variance of $\mathbf{Y}(1)$?

3. (45 points) Consider a binary baseband digital communication system operating on an AWGN channel with transmitted signals given by

$$s_0(t) = A \bullet p_T(t) \quad \text{and} \quad s_1(t) = A \bullet (t/T) \bullet p_T(t).$$

We consider the error probability performance for three receivers for this signal set.

(a) Suppose that the receiver consists of a linear filter with impulse response $h(t) = A \bullet p_T(t)$, a sampler at time T, and the minimax threshold is used. Note that $h(t) = s_0(t)$. What is the minimax error probability P_e ? Call this number $P_{e,\text{part(a)}}$.

(b) Now suppose that the receiver filter is $h(t) = A \bullet (t/T) \bullet p_T(t)$, while the sampling time remains T as before, and the appropriate minimax threshold is used. Note also that $h(t) = s_1(t)$. What is the value of the minimax error probability P_e? Call this number $P_{e,\text{part(b)}}$.

(c) What is the optimum minimax error probability achievable with signals $s_0(t)$ and $s_1(t)$? You are not asked to compute the impulse response of the matched filter, or the minimax threshold, or … for this receiver. I just want to know P_e. Call this number $P_{e,\text{part(c)}}$

(d) Compare probabilities $P_{e,\text{part(a)}}$, $P_{e,\text{part(b)}}$ and $P_{e,\text{part(c)}}$ to determine which is the smallest.

(e) Do ONE of the following three parts as appropriate:

(i) If you found that $P_{e,\text{part(a)}} < P_{e,\text{part(c)}}$ **or** $P_{e,\text{part(b)}} < P_{e,\text{part(c)}}$ (**or** both!!) explain why the matched filter receiver of part (c), which is supposed to be optimum, has poorer performance than the receiver of part (a) or the receiver of part (b).

(ii) If you found that $P_{e,\text{part(a)}} > P_{e,\text{part(c)}}$ **and** $P_{e,\text{part(b)}} = P_{e,\text{part(c)}}$, explain why a receiver with filter $h(t) = s_1(t)$ has better performance than a receiver with filter $h(t) = s_0(t)$.

(iii) If you found that $P_{e,\text{part(a)}} \ge P_{e,\text{part(c)}}$ **and** $P_{e,\text{part(b)}} > P_{e,\text{part(c)}}$, draw a neat sketch of the signals $s_0(t)$ and $s_1(t)$, and the impulse response of the matched filter of the receiver of part (c), all on the same axes (for ease of comparison).

4. (40 points) Let $s_0(t)$ and $s_1(t)$ denote two signals of finite energies E_0 and E_1 respectively.

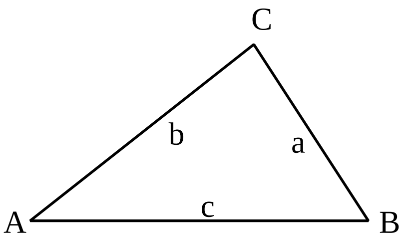
(a) What is the optimum minimax error probability achievable with these signals in an AWGN channel?

Now suppose that $s_0(t)$ and $s_1(t)$ are represented as points $\mathbf{s}_0 = (c_{00},c_{01})$ and $\mathbf{s}_1 = (c_{10},c_{11})$ in a signal space with orthonormal basis functions $\{\psi_0(t), \psi_1(t)\}$. Do **NOT** assume that $\psi_0(t)$ and $\psi_1(t)$ were obtained via a Gram-Schmidt orthonormalization procedure, that is, do **not** assume that $s_0(t) = c_{00}\bullet\psi_0(t)$ and thus $c_{01} = 0$; none of c_{00}, c_{01}, c_{10},and c_{11} is necessarily equal to 0.

(b) Let d_{01} denote the distance between $\mathbf{s}_0$ and $\mathbf{s}_1$. Express the optimum minimax error probability in terms of d_{01}, N_0 (and components c_{00}, c_{01}, c_{10}, c_{11} of $\mathbf{s}_0$ and $\mathbf{s}_1$ if needed).

(c) Let d_0 and d_1 denote the distances of $\mathbf{s}_0$ and $\mathbf{s}_1$ respectively from the origin $\mathbf{0} = (0,0)$. Let θ denote the angle between the straight line through $\mathbf{s}_0$ and $\mathbf{0}$ and the straight line through $\mathbf{s}_1$ and $\mathbf{0}$. What is $\cos\theta$? Express your answer in two different forms: in terms of d_0 and d_1 (and c_{00}, c_{01}, c_{10}, c_{11} if needed), and in terms of E_0 and E_1 (and c_{00}, c_{01}, c_{10}, c_{11} if needed)

(d) In a triangle, any side can be expressed in terms of the other two sides and the angle between these other sides. For example, in the triangle ABC shown below,



$c^2 = a^2 + b^2 - 2\bullet a\bullet b\bullet\cos C$, and similarly for sides a and b. Use this result to derive a relationship between d_{01}, d_0, and d_1 and thus show that the argument of $Q(\bullet)$ that you obtained in part (b) is the same as the argument of $Q(\bullet)$ that you obtained in part (a).

(e) What choice(s) of $\mathbf{s}_0$ and $\mathbf{s}_1$ will minimize the optimum minimax error probability and what is this minimum value of the optimum minimax error probability? Do not forget that the signals are constrained to have energies E_0 and E_1 respectively. Drawing a sketch will help you think through this problem.

(f) Let us return to the case of *arbitrary* points $\mathbf{s}_0$ and $\mathbf{s}_1$ at distance d_{01} apart, and representing signals $s_0(t)$ and $s_1(t)$ of energies E_0 and E_1 respectively. Bear in mind that $\mathbf{s}_0$ and $\mathbf{s}_1$ are not necessarily in the locations that you described in your answer to part (e).

Consider a modified receiver that works as follows. If the received vector $\mathbf{r} = (r_0,r_1)$ is within distance $d_{01}/2$ of one of the signal points $\mathbf{s}_0$ and $\mathbf{s}_1$, the receiver decides that that signal was transmitted. If neither signal point is within distance $d_{01}/2$ of $\mathbf{r}$, the receiver asks the transmitter to re-send the signal. For i = 0, 1, let Γ_i denote the set of all $\mathbf{r}$ for which the decision is that $\mathbf{s}_i$ was transmitted, and note that Γ_0 and Γ_1 are disjoint sets but their union is **not** the entire plane.

(i) Draw a sketch illustrating the decision regions Γ_0 and Γ_1.

(ii) Show that $P(\mathbf{r} \quad \Gamma_0 \mid s_0(t)$ transmitted), the probability that the receiver correctly decides that $s_0(t)$ was transmitted when $s_0(t)$ was in fact transmitted, equals $1 - \exp(-d_{01}^2/4N_0)$.