Thursday, April 5, 2001

1:00 p.m. — 2:20 p.m.

161 Everitt Laboratory

Instructions:

There are **three problems** on this examination worth a total of 125 points.

One page of notes allowed. No other notes, books, tables of integrals, and calculators/personal computers permitted.

Show all your work in the exam booklet provided. Answers without appropriate justification will receive no credit.

Notation and (possibly) useful formulas

$\Phi(x) =$ cumulative probability distribution function for zero-mean unit-variance (i.e., standard) Gaussian random variable

$Q(x) = 1 - \Phi(x)$

$P_{e,i} =$ probability of error given that signal $s_i(t)$ was transmitted

AWGN denotes additive white Gaussian noise with (two-sided) power spectral density $N_0/2$. This random process is independent of the choice of transmitted signal. The mean of this process is 0.

$$\text{rect}(t) = \begin{cases} 1 & \text{if } -1/2 \le t \le 1/2, \\ 0, & \text{otherwise.} \end{cases} \qquad p_T(t) = \begin{cases} 1, & \text{if } 0 \le t < T, \\ 0, & \text{otherwise.} \end{cases}$$

$$X(t) \leftrightarrow x(-f) \qquad x(\alpha t) \leftrightarrow |\alpha|^{-1} \cdot X(\alpha^{-1} f)$$

$$x^*(t) \leftrightarrow X^*(-f) \qquad x^*(-t) \leftrightarrow X^*(f)$$

$$x(t-t_0) \leftrightarrow X(f) \cdot \exp(-j2\pi f t_0) \qquad x(t) \cdot \exp(j2\pi f_0 t) \leftrightarrow X(f-f_0)$$

$$x(t) \cdot \cos(2\pi f_0 t) \leftrightarrow (1/2)[X(f+f_0) + X(f-f_0)] \qquad x(t) \cdot \sin(2\pi f_0 t) \leftrightarrow (j/2)[X(f+f_0) - X(f-f_0)]$$

$$\frac{d}{dt} x(t) \leftrightarrow (j2\pi f) \cdot X(f) \qquad \int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow (j2\pi f)^{-1} \cdot X(f) + X(0) \cdot \delta(f)/2$$

$$x * y = \int_{-\infty}^{\infty} x(\tau) \cdot y(t-\tau) d\tau \leftrightarrow X(f) \cdot Y(f) \qquad x(t) \cdot y(t) \leftrightarrow \int_{-\infty}^{\infty} X(\lambda) \cdot Y(f-\lambda) d\lambda$$

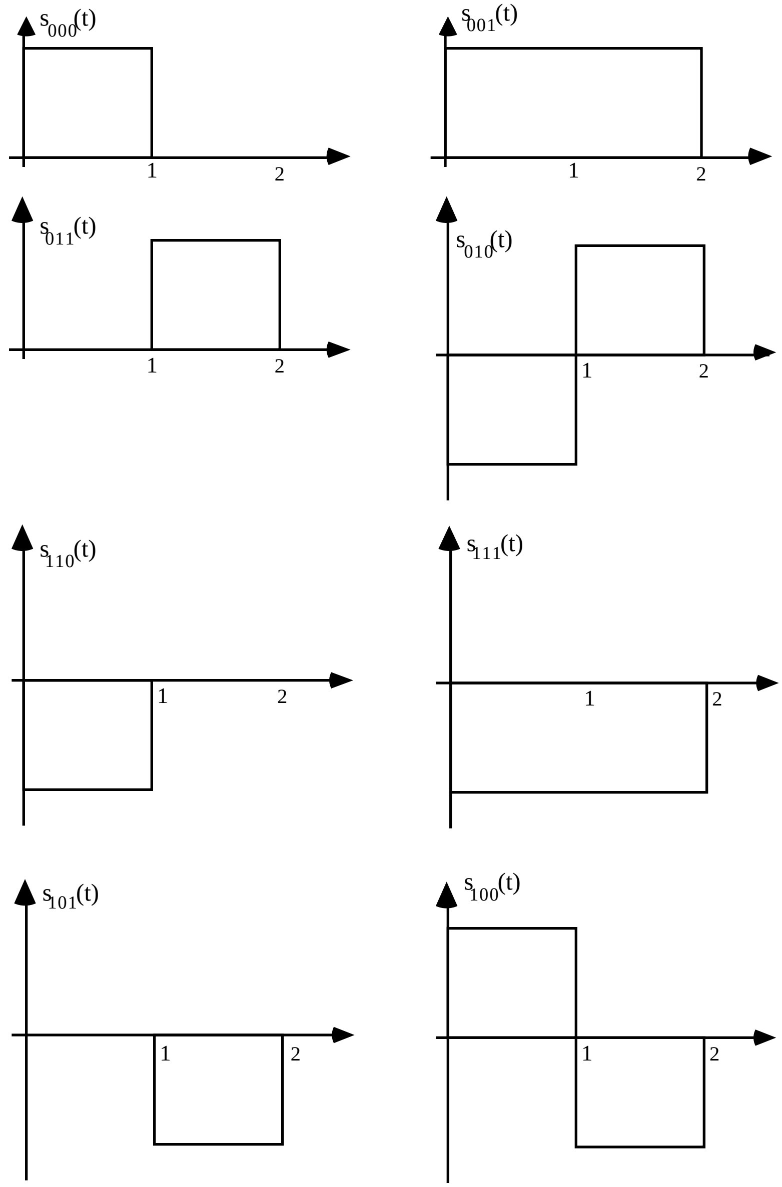
$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \qquad \int_{-\infty}^{\infty} x(t) y^*(t) dt = \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

$$R_{x,y}(\tau) = \int_{-\infty}^{\infty} x(t+\tau) \cdot y^*(t)\, dt \qquad R_{x,y}(\tau) \leftrightarrow S_{x,y}(f) = X(f) Y^*(f)$$

$$\text{rect}(t/T) \leftrightarrow T \cdot \text{sinc}(fT) \qquad W \cdot \text{sinc}(Wt) \leftrightarrow \text{rect}(f/W)$$

where $\text{sinc}(z) = \frac{\sin \pi z}{\pi z}$ is the *sinc function*

1. **(55 points)** Consider a communication system operating over a AWGN channel with two-sided power spectral density $N_0/2$ using the 8 equally likely signals shown below.



Each pulse is of amplitude $\pm\sqrt{E}$.

(a) What is the average energy *per bit* for this signal set?

(b) Sketch the signal constellation in signal space with respect to a suitable orthonormal basis, and show the maximum-likelihood decision regions.

In the next three parts, you are asked to calculate various error probabilities. If you cannot find the exact answer, give upper and lower bounds (tighter than 1 and 0 respectively!!) on the desired probabilities.

(c) What is the probability of a symbol error given that $s_{001}(t)$ is transmitted?

(d) What is the probability of a symbol error given that $s_{000}(t)$ is transmitted?

(e) What is the average probability of symbol error?

In the next two parts, you are again asked to calculate various error probabilities. Exact answers are expected: no upper and lower bounds or approximations, please.

(f) Assume that $s_{000}(t)$ was transmitted, and let p_{ijk} denote the conditonal probability that the receiver decides that signal $s_{ijk}(t)$ was transmitted. What is the average probability of bit error given that $s_{000}(t)$ was transmitted? Express your answer in terms of the p_{ijk}'s. **Do not** attempt to express the probabilities p_{ijk} in terms of Q(•).

(g) A 4-ary communication system operates as follows.

During the time interval [4n, 4n+2), the signals $s_{000}(t)$, $s_{011}(t)$, $s_{110}(t)$, and $s_{101}(t)$ are used to transmit 00, 01, 11, and 10 respectively. Note that these signals are in the left column of the diagram on the previous page, and that the transmitted bits are just the first two bits of the subscript

During the time interval [4n+2, 4n+4), the signals $s_{001}(t)$, $s_{010}(t)$, $s_{111}(t)$, and $s_{100}(t)$ are used to transmit 00, 01, 11, and 10 respectively. Note that these signals are in the right column of the diagram on the previous page, and that the transmitted bits are just the first two bits of the subscript.

What are the average bit error probabilities during the two time intervals [4n, 4n+2) and [4n+2, 4n+4)? Are the errors in the two bits transmitted during each two-second time interval independent events? Are the errors in the bits transmitted in different two-second time intervals independent events? Explain.

2. **(25 points)** At time t = (k+1)T, an optimum receiver for binary antipodal PSK signals in AWGN produces the decision statistic $\mathbf{Z}_k = A(-1)^{b_k} + \mathbf{N}_k$ where b_k {0,1} denotes the k-th transmitted bit, and $\mathbf{N}_k$ is a zero-mean Gaussian random variable with variance σ^2. The b_k's and $\mathbf{N}_k$'s are mutually independent random variables.

(a) Let $\hat{b}_k$ {0,1} denote the receiver's decision about the value of the k-th transmitted bit b_k. The receiver compares $\mathbf{Z}_k$ to 0 and sets $\hat{b}_k = 0$ if $\mathbf{Z}_k > 0$ and $\hat{b}_k = 1$ if $\mathbf{Z}_k \le 0$. What is $P(\hat{b}_k \neq b_k)$? Express your answer in terms of the parameters A and σ.

From here onwards, denote the error probability that you found in part (a) by p. DO NOT use the expression involving A and σ that you obtained in part (a).

(b) Assume that the decision regarding $\hat{b}_k$ is made as described above in part (a). Now suppose, however, that at the transmitter, differential encoding is used so that the k-th **transmitted** bit b_k is related to the k-th **data** bit a_k as

$$b_k = a_k \quad b_{k-1}$$

where denotes the Exclusive-OR operation. The receiver decides that the data bit has value $\hat{a}_k$ where

$$\hat{a}_k = \hat{b}_k \quad \hat{b}_{k-1}$$

What is $P(\hat{a}_k \neq a_k)$? Express your answer in terms of p.

No partial credit for an incorrect answer in terms of A and σ.

(c) Assume, as in part (b), that differential encoding is being used at the transmitter. The receiver, however, does not use $\mathbf{Z}_k$ to decide the value of $\hat{b}_k$ as described in part (a). Instead, the receiver stores the value of the decision statistic $\mathbf{Z}_{k-1}$ (which it produced at time t = kT) in its memory during the time interval (kT, (k+1)T] during which it is computing $\mathbf{Z}_k$. When the receiver determines $\mathbf{Z}_k$ at t = (k+1)T, it sets

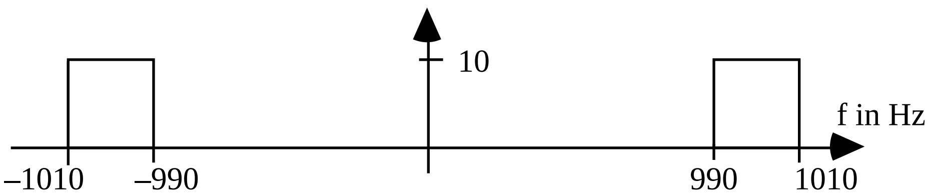
$$\hat{a}_k = \begin{cases} 0, & \text{if } (\mathbf{Z}_k + \mathbf{Z}_{k-1})^2 > (\mathbf{Z}_k - \mathbf{Z}_{k-1})^2, \\ 1, & \text{if } (\mathbf{Z}_k + \mathbf{Z}_{k-1})^2 \le (\mathbf{Z}_k - \mathbf{Z}_{k-1})^2. \end{cases}$$

(Then the receiver throws away $\mathbf{Z}_{k-1}$ and stores $\mathbf{Z}_k$ instead during the time interval ((k+1)T, (k+2)T] for use in the decision at t = (k+2)T, etc.)

Note that the receiver makes a decision about the data bit a_k directly without computing the transmitted bit b_k at all!

Find $P(\hat{a}_k \ne a_k)$ for this receiver. Express your answer in terms of p.

3. **(45 points)** A zero-mean wide-sense-stationary narrowband Gaussian random process $\{\mathbf{N}(t)\}$ has power spectral density $S_{\mathbf{N}}(f)$ as shown below (not to scale).



(a) Express $P\{\mathbf{N}(t) > 20\}$ in terms of Q(•).

The quadrature components $\{\mathbf{X}(t)\}$ and $\{\mathbf{Y}(t)\}$ of $\{\mathbf{N}(t)\}$ with respect to carrier frequency 1 kHz are related to the narrowband process by

$$\mathbf{N}(t) = \mathbf{X}(t)\cdot\cos(2\pi(1000)t) - \mathbf{Y}(t)\cdot\sin(2\pi(1000)t).$$

(b) Find the autocorrelation function $R_{\mathbf{X}}(t)$ of the random process $\{\mathbf{X}(t)\}$.

(c) Are the *random variables* $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ uncorrelated and/or independent? Explain.
Are the *processes* $\{\mathbf{X}(t)\}$ and $\{\mathbf{Y}(t)\}$ uncorrelated and/or independent? Explain.

We can also write $\mathbf{N}(t) = \hat{\mathbf{X}}(t)\cdot\cos(2\pi(990)t) - \hat{\mathbf{Y}}(t)\cdot\sin(2\pi(990)t)$ where $\{\hat{\mathbf{X}}(t)\}$ and $\{\hat{\mathbf{Y}}(t)\}$ are the quadrature components with respect to carrier frequency 0.99 kHz.

(d) What is the autocorrelation function of the random process $\{\hat{\mathbf{X}}(t)\}$?

(e) Are the *random variables* $\hat{\mathbf{X}}(t)$ and $\hat{\mathbf{Y}}(t)$ uncorrelated? Explain.
Are the *processes* $\{\hat{\mathbf{X}}(t)\}$ and $\{\hat{\mathbf{Y}}(t)\}$ uncorrelated? Explain.