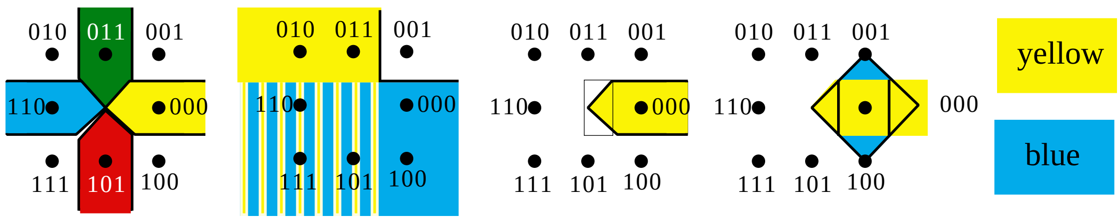
1.(a) Four of the signals have energy E while four have energy 2E. Hence, the total energy is 12E, the average symbol energy is 1.5E, and since there are three bits in each symbol, the average energy per bit is 0.5E.

(b) The obvious orthonormal basis is $\{p_1(t), p_1(t-1)\}$, and with respect to these, the signals form an 8-ary QAM signal set with coordinates (α,β) where α and β take on the three values $\pm\sqrt{E}$ and 0 (except (0,0)). The signal constellation and maximum-likelihood decision regions are shown in the first figure from the left in the illustration below, where the minimum distance between signal points is $\sqrt{E}$.



(c) The decision region for $s_{001}(t)$ is the same as that for a corner point in QAM. Hence, the probability of a symbol error is $P_{c,\text{QAM corner point}} = 2Q(x) - Q^2(x)$ where $x = \sqrt{E/2N_0} = d/\sqrt{2N_0}$. Doing it the hard way, $P_{e,001}$ can be expressed as
P{**r** yellow region or **r** blue region} (see the second figure from the left in the illustration above)
= P{**r** yellow region} + P{**r** blue region} – P{**r** intersection, i.e. region with yellow/blue stripes}
$= Q(x) + Q(x) - Q^2(x) = 2Q(x) - Q^2(x)$ by independence. If you are reading this on a greyscale printed page, the shadings indicating yellow and blue are as shown in the rightmost figure above.

Lower and upper bounds $Q(x) < P_{e,001} < 2Q(x)$ are acceptable for reduced credit.

(d) The decision region for $s_{000}(t)$ consists of the decision region for an edge point in QAM, plus a triangular endpiece (yellow region in the third figure from the left in the illustration above.) Hence, $P_{c,000} > P_{c,\text{QAM edge point}}$ and therefore, $P_{e,000} < P_{e,\text{QAM edge point}} = 3Q(x) - 2Q^2(x)$. A weak upper bound on $P_{c,000}$ is obtained by extending the triangular endpiece into a rectangle as shown above and noting that P{**r** elongated strip} $= [1 - 2Q(x)]\bullet[1 - Q(2x)]$ because the point 110 is at distance $2\sqrt{E}$ from 000. Hence, $P_{e,000} > 2Q(x) + Q(2x) - 2Q(x)Q(2x)$. These upper and lower bounds are acceptable for reduced credit. The *exact* value of $P_{e,000}$ can, in fact, be calculated by "completing the square" as illustrated in the fourth figure from the left in the illustration above. But, P{**r** in square} $= P_{c,\text{QAM interior point}}$ is of the form $[1 - 2Q(y)]^2$ where y depends on the grid spacing, and so we get
P{**r** triangular endpiece} = 0.25•[P{**r** large square} – P{**r** small square}]
$= 0.25\bullet[1 - 2Q(x\sqrt{2})]^2 - 0.25\bullet[1 - 2Q(x)]^2 = Q(x) - Q(x\sqrt{2}) - Q^2(x) + Q^2(x\sqrt{2})$. It follows that
$P_{c,000} = P_{c,\text{QAM edge point}}$ + P{**r** triangular endpiece}, and hence
$P_{e,000} = 3Q(x) - 2Q^2(x) - [Q(x) - Q(x\sqrt{2}) - Q^2(x) + Q^2(x\sqrt{2})] = 2Q(x) - Q^2(x) + Q(x\sqrt{2}) - Q^2(x\sqrt{2})$
exactly! The same result can also be obtained via the approach used for the corner point error probability in 32-CROSS (cf. Solutions to Problem 1 of Problem Set #8). Note also that $P_{e,000}$ is *only slightly larger* than $P_{e,001} = 2Q(x) - Q^2(x)$.

(e) Since there are four corner points and four edge points, it follows that

$$P_{c,\text{symbol}} = 0.5\bullet[2Q(x) - Q^2(x) + 2Q(x) - Q^2(x) + Q(x\sqrt{2}) - Q^2(x\sqrt{2})]$$
$$= 2Q(x) - Q^2(x) + 0.5\bullet[Q(x\sqrt{2}) - Q^2(x\sqrt{2})]$$

(f) Conditioned on $s_{000}(t)$ being transmitted,

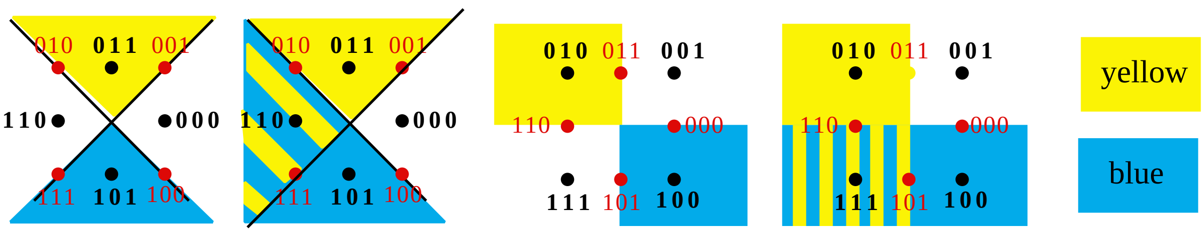
P{1st bit in error} = P{receiver decides that one of $s_{100}(t)$, $s_{101}(t)$, $s_{110}(t)$, $s_{111}(t)$ was transmitted}
$= p_{100} + p_{101} + p_{110} + p_{111}$

P{2nd bit in error} = P{receiver decides that one of $s_{010}(t)$, $s_{011}(t)$, $s_{110}(t)$, $s_{111}(t)$ was transmitted}
$= p_{010} + p_{011} + p_{110} + p_{111}$

P{3rd bit in error} = P{receiver decides that one of $s_{001}(t)$, $s_{101}(t)$, $s_{011}(t)$, $s_{111}(t)$ was transmitted}
$= p_{001} + p_{101} + p_{011} + p_{111}$

Hence, conditioned on $s_{000}(t)$ being transmitted, the average bit error probability $\bar{P}_b(000)$
$= (1/3)\bullet[\{p_{100} + p_{101} + p_{110} + p_{111}\} + \{p_{010} + p_{011} + p_{110} + p_{111}\} + \{p_{001} + p_{101} + p_{011} + p_{111}\}]$
$= (1/3)\bullet[\{p_{100} + p_{010} + p_{001}\} + 2\bullet\{p_{110} + p_{011} + p_{101}\} + 3\bullet p_{111}]$
Note that each p_{ijk} is multiplied by the *Hamming weight* of ijk, (i.e., number of nonzero entries in ijk) or more generally, by the *Hamming distance* between ijk and xyz (i.e. number of bit positions where ijk and xyz differ) if $s_{xyz}(t)$ is transmitted (instead of $s_{000}(t)$). Note that if we replace each such coefficient by 3, the right side increases to $P_{e,000}$, while if we replace each such coefficient by 1, the right side decreases to $(1/3)\bullet P_{e,000}$. Thus, $(1/3)\bullet P_{e,000} < \bar{P}_b(000) < P_{e,000}$. A similar result applies to each $s_{xyz}(t)$. The general result that follows from all this is that in 2^k-ary communication systems, the average symbol error probability $\bar{P}_e$ and the average bit error probability $\bar{P}_b$ satisfy the relationship $(1/k)\bullet\bar{P}_e < \bar{P}_b < \bar{P}_e$.

(g) The signal sets and decision regions are as shown. Note that both signal sets are QAM signal sets with minimum distances $\sqrt{2E}$ and $2\sqrt{E}$ respectively.



For the first set, assuming that $s_{000}(t)$ is transmitted, both bits have the same error probabiity $P_{e,b}$ = P{**r** yellow region} = P{**r** blue region} = $Q(x\sqrt{2})$. The same result is obtained when other signals are transmitted. Also, P{both bits in error} = P{**r** yellow/blue striped region} = $[Q(x\sqrt{2})]^2$ showing that the bit errors are independent. Exactly the same argument applies to the second set as well, and we have that $P_{e,b}$ = P{**r** yellow region} = P{**r** blue region} = Q(2x). The same result is obtained when other signals are transmitted. Also, P{both bits in error} = P{**r** yellow/blue striped region} = $[Q(2x)]^2$ showing that the bit errors are independent.
Finally, the noise variables at time 4n+2 are obtained by integrating AWGN over [4n,4n+2) while those at time 4n+4 are obtained via integration over [4n+2,4n+4). Hence, these are independent, and the probability that both of them exceed a certain threshold is simply the product of the probabilities that each of them exceeds that threshold, i.e. bit errors at different sampling instants are independent.

2.(a) If $b_k = 0$, $\mathbf{Z}_k$ is $N(A,\sigma^2)$ and hence $P(\mathbf{Z}_k \le 0 \mid b_k = 0) = Q(A/\sigma)$. Similarly, if $b_k = 1$, $\mathbf{Z}_k$ is $N(-A,\sigma^2)$ and hence $P(\mathbf{Z}_k > 0 \mid b_k = 1) = Q(A/\sigma)$, and hence $P(E) = Q(A/\sigma) = p$.

(b) $P(\hat{a}_k \ne a_k) = P(\hat{b}_k \ne b_k, \hat{b}_{k-1} = b_{k-1}) + P(\hat{b}_k = b_k, \hat{b}_{k-1} \ne b_{k-1})$
$= P(\hat{b}_k \ne b_k)P(\hat{b}_{k-1} = b_{k-1}) + P(\hat{b}_k = b_k)P(\hat{b}_{k-1} \ne b_{k-1})$ by independence of the events
$= p(1-p) + (1-p)p = 2p - 2p^2 = 2Q(A/\sigma) - 2Q^2(A/\sigma)$.

(c) $(\mathbf{Z}_k + \mathbf{Z}_{k-1})^2 > (\mathbf{Z}_k - \mathbf{Z}_{k-1})^2$ implies that $\mathbf{Z}_k\bullet\mathbf{Z}_{k-1} > 0$. But, if $a_k = 0$, then it must be that
either $b_k = b_{k-1} = 0$ in which case $\mathbf{Z}_k$ and $\mathbf{Z}_{k-1}$ both are $N(A,\sigma^2)$
or $b_k = b_{k-1} = 1$ in which case $\mathbf{Z}_k$ and $\mathbf{Z}_{k-1}$ both are $N(-A,\sigma^2)$.
In either case, $P(\mathbf{Z}_k\bullet\mathbf{Z}_{k-1} > 0) = P(\mathbf{Z}_k > 0, \mathbf{Z}_{k-1} > 0) + P(\mathbf{Z}_k < 0, \mathbf{Z}_{k-1} < 0)$
$= P(\mathbf{Z}_k > 0)P(\mathbf{Z}_{k-1} > 0) + P(\mathbf{Z}_k < 0)P(\mathbf{Z}_{k-1} < 0) = (1-p)^2 + p^2$. A similar analysis holds if $a_k = 1$.
Hence, the probability of a correct decision is $(1-p)^2 + p^2 = 1 - 2p + 2p^2$, and the probability of error is $2p - 2p^2$ just as in part (b).

Generally speaking, *soft-decision* receivers are those that use real number decision statistics to determine the data bit(s) directly. In contrast, *hard-decision* receivers quantize the matched filter outputs into bits $\hat{b}_{k-1}$ and $\hat{b}_k$ and use digital logic to determine the data bit(s). As a rule of thumb, soft-decision receivers provide better error probability performance, but are more complicated to implement than hard-decision receivers. But, for plain vanilla differentially encoded PSK, there is no advantage to soft-decision receivers. Bummer!

3.(a) $E[\mathbf{N}(t)] = 0$ since this is a narrowband (Gaussian) random process.
$\text{var}(\mathbf{N}(t)) = R_{\mathbf{N}}(0)$ = area under $S_{\mathbf{N}}(f) = 10\times(20+20) = 400$, i.e. $\sigma = 20$. Hence, $P\{\mathbf{N}(t) > 20\} = Q(1)$

(b) $R_{\mathbf{N}}(t) = 2\int_0^\infty S_{\mathbf{N}}(f)\cdot\cos(2\pi ft)df = 20\cdot\frac{\sin(2\pi ft)}{2\pi t}\Big|_{990}^{1010} = \frac{20}{2\pi t}\cdot[\sin(2\pi(1010)t) - \sin(2\pi(990)t)]$

$= \frac{20}{\pi t}\cdot[\cos(2\pi(1000)t)\cdot\sin(2\pi(10t))] = 400\cdot\text{sinc}(20t)\cdot\cos(2\pi(1000)t)$. Since we know that $R_{\mathbf{N}}(t) = R_{\mathbf{X}}(t)\cdot\cos(2\pi(1000)t) - R_{\mathbf{X},\mathbf{Y}}(t)\cdot\sin(2\pi(1000)t)$, we deduce that $R_{\mathbf{X}}(t) = 400\cdot\text{sinc}(20t)$ and $R_{\mathbf{X},\mathbf{Y}}(t) = 0$. Alternatively, note that $S_{\mathbf{X}}(f) = \begin{cases} S_{\mathbf{N}}(f-1000) + S_{\mathbf{N}}(f+1000), & |f| < 1000, \\ 0, & \text{otherwise,} \end{cases}$ and

$S_{\mathbf{X},\mathbf{Y}}(f) = \begin{cases} j[S_{\mathbf{N}}(f-1000) - S_{\mathbf{N}}(f+1000)], & |f| < 1000, \\ 0, & \text{otherwise,} \end{cases}$ giving us that $S_{\mathbf{X}}(f) = 20\cdot\text{rect}(f/20)$ with inverse Fourier transform $R_{\mathbf{X}}(t) = 20\cdot(20\cdot\text{sinc}(20t)) = 400\cdot\text{sinc}(20t)$, while $S_{\mathbf{X},\mathbf{Y}}(f) = 0$ and hence $R_{\mathbf{X},\mathbf{Y}}(t) = 0$.

(c) For any given time instant t, the *random variables* $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ are uncorrelated for all narrowband processes. Since $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ are (jointly) Gaussian, they are also independent. *In this case,* the random processes $\{\mathbf{X}(t)\}$ and $\{\mathbf{Y}(t)\}$ have zero crosscorrelation function and hence are uncorrelated processes. Since the processes are (jointly) Gaussian processes, they are also independent processes.

(d) Proceeding as in part (b), $S_{\hat{\mathbf{X}}}(f) = \begin{cases} S_{\mathbf{N}}(f-990) + S_{\mathbf{N}}(f+990), & |f| < 990, \\ 0, & \text{otherwise,} \end{cases}$ and

$S_{\hat{\mathbf{X}},\hat{\mathbf{Y}}}(f) = \begin{cases} j[S_{\mathbf{N}}(f-990) - S_{\mathbf{N}}(f+990)], & |f| < 990, \\ 0, & \text{otherwise.} \end{cases}$ Hence, $S_{\hat{\mathbf{X}}}(f) = 10\cdot\text{rect}(f/40)$ with inverse Fourier transform $R_{\hat{\mathbf{X}}}(t) = 10\cdot(40\cdot\text{sinc}(40t)) = 400\cdot\text{sinc}(40t)$.

For any given time instant t, the *random variables* $\hat{\mathbf{X}}(t)$ and $\hat{\mathbf{Y}}(t)$ are uncorrelated for all narrowband processes. Since $\{\hat{\mathbf{X}}(t)\}$ and $\{\hat{\mathbf{Y}}(t)\}$ are (jointly) Gaussian, they are also independent. *In this case,* the random processes $\{\hat{\mathbf{X}}(t)\}$ and $\{\hat{\mathbf{Y}}(t)\}$ do not have zero cross-spectral density, and hence they do not have zero crosscorrelation function. Hence they are correlated (and therefore, dependent) processes.