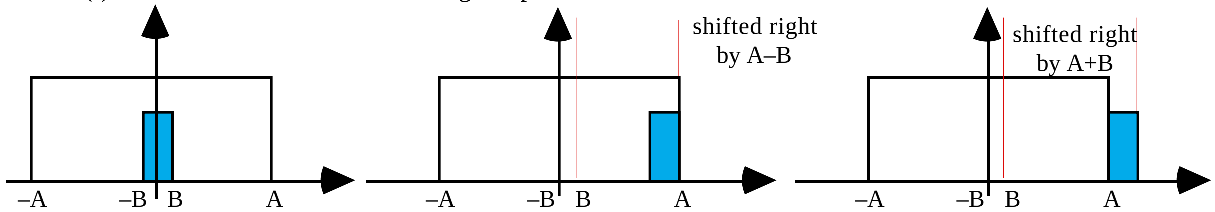
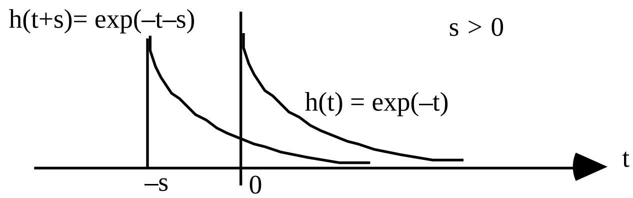
1. S(f) is the convolution of two rectangular pulses as shown below.



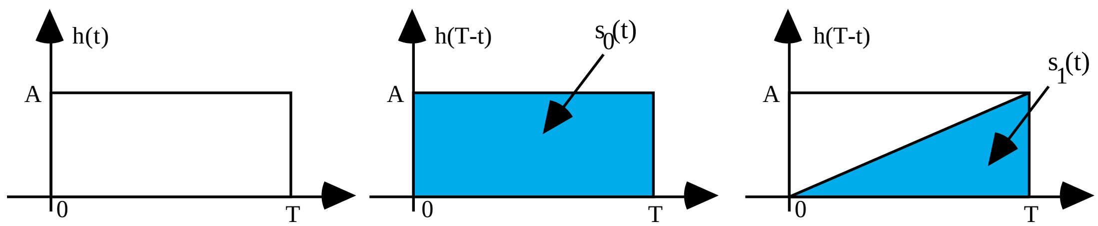
We readily get that the linear decrease in S(f) starts at "shift" A–B = 10 and ends at "shift" A+B = 12.2 giving A = 11.1 and B = 1.1, that is, the two pulses are of the form a•rect(f/22.2) and b•rect(f/2.2) (Why did I use 2A and 2B in the argument?). Since convolution in the frequency domain corresponds to multiplication in the time domain, s(t) = c•sinc(2.2t)•sinc(22.2t) for some constant c. Now, s(0) = c = area under S(f) is just (12.2 + 10)•(5/111) = 22.2•(5/111) = 1, and hence s(t) = sinc(2.2t)•sinc(22.2t).

2. The output is also a zero-mean WSS process, and hence $\mathrm{var}(\mathbf{Y}(1)) = \mathrm{var}(\mathbf{Y}(t)) = R_{\mathbf{Y}}(0) = \int_{-\infty}^{\infty} g(t)R_{\mathbf{X}}(t)\,dt$.



For s > 0, we have that $g(s) = \int_{-\infty}^{\infty} h(t+s)h(t)\,dt = \int_{0}^{\infty} \exp(-t-s)\exp(-t)\,dt = (1/2)\cdot\exp(-s)$. Since g(•) is the autocorrelation function of h(t), and thus an even function, we deduce that for s < 0, g(s) = (1/2)•exp(s), i.e. $g(t) = (1/2)\cdot\exp(-|t|)$ for $-\infty < t < \infty$. Thus, $R_{\mathbf{Y}}(0) = \int_{-\infty}^{\infty} g(t)R_{\mathbf{X}}(t)\,dt = 2\int_{0}^{\infty}(1/2)\cdot\exp(-t)\cdot\exp(-2t)\,dt = 1/3$.

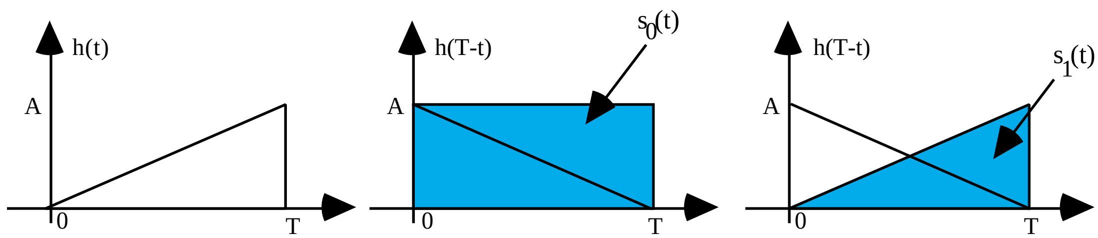
3.(a) The signal outputs from the filter at the sampling instant T are $\int_0^T A^2\,dt = A^2T$ and $\int_0^T A^2(t/T)\,dt = A^2T/2$.



The noise variance at the filter output is $\sigma^2 = (N_0/2)\int_0^T h^2(t)\,dt = (N_0/2)\int_0^T A^2\,dt = N_0A^2T/2$.

Hence, $P_{e,\text{part(a)}} = Q\left(\frac{\text{difference in signal outputs}}{2\sigma}\right) = Q\left(\frac{A^2T - A^2T/2}{2\sigma}\right) = Q\left(\sqrt{\frac{A^2T}{8N_0}}\right)$

(b) Now, the signal outputs at t = T are $\int_0^T A^2((T-t)/T)\,dt = A^2T/2$ (cf. part (a)) and $\int_0^T A^2[(T-t)/T](t/T)\,dt) = A^2T/6$.



The noise variance at the filter output is $\sigma^2 = (N_0/2)\int_0^T h^2(t)dt = (N_0/2)\int_0^T [A(T-t)/T]^2 dt = N_0A^2T/6$.

Hence, $P_{e,part(b)} = Q\left(\frac{\text{difference in signal outputs}}{2\sigma}\right) = Q\left(\frac{A^2T/2 - A^2T/6}{2\sigma}\right) = Q\left(\sqrt{\frac{A^2T}{6N_0}}\right)$

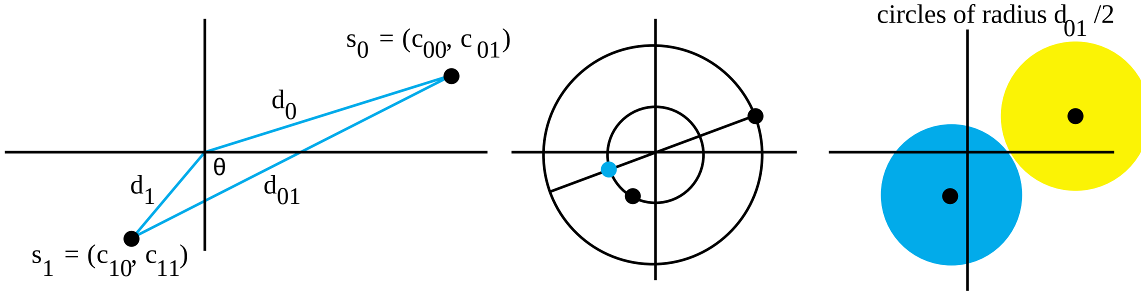
(c) The difference signal $s_0(t) - s_1(t) = A[(T-t)/T]\bullet p_T(t)$ has energy $\int_0^T [A(T-t)/T]^2 dt = A^2T/3$ (cf. part (b)!)

Hence, $P_{e,part(c)} = Q\left(\sqrt{\frac{\text{energy in difference signal}}{2N_0}}\right) = Q\left(\sqrt{\frac{A^2T}{6N_0}}\right)$

(d) Since $Q(\bullet)$ is a decreasing function of its argument, $P_{e,part(a)} > P_{e,part(b)} = P_{e,part(c)}$

(e)(ii) The reason that the receiver with filter $h(t) = s_1(t)$ has the same performance as the matched filter receiver (and better performance than the receiver with filter $h(t) = s_0(t)$) is that the matched filter serendipitiously just happens to be $s_1(t)$!!. Note that the matched filter for sampling time $t = T$ has impulse response given by $s_0(T-t) - s_1(T-t) = s_1(t)$ *for this signal set.*

4.(a) $P_e = Q\left(\sqrt{\frac{E_0 + E_1 - 2\, \mathbf{s}_0,\mathbf{s}_1}{2N_0}}\right)$ **(b)** $P_e = Q\left(\frac{d_{01}}{\sqrt{2N_0}}\right)$



(c) The inner product (dot-product) $\mathbf{s}_0,\mathbf{s}_1$ is given by $c_{00}c_{10} + c_{01}c_{11}$. We know that the inner product can also be expressed as $\mathbf{s}_0,\mathbf{s}_1 = |\mathbf{s}_0|\bullet|\mathbf{s}_1|\bullet\cos\theta$ where $|\mathbf{s}_i| = d_i = \sqrt{(c_{i0})^2 + (c_{i1})^2} = \sqrt{E_i}$.

Hence, $\cos\theta = \frac{c_{00}c_{10} + c_{01}c_{11}}{\sqrt{(c_{00})^2 + (c_{01})^2}\sqrt{(c_{10})^2 + (c_{11})^2}} = \frac{c_{00}c_{10} + c_{01}c_{11}}{d_0d_1} = \frac{c_{00}c_{10} + c_{01}c_{11}}{\sqrt{E_0E_1}} = \frac{\mathbf{s}_0,\mathbf{s}_1}{\sqrt{E_0E_1}}$

(d) $(d_{01})^2 = (d_0)^2 + (d_1)^2 - 2d_0d_1\cos\theta = E_0 + E_1 - 2\sqrt{E_0}\sqrt{E_1}\cos\theta = E_0 + E_1 - 2\, \mathbf{s}_0,\mathbf{s}_1$.

(e) The signal points must lie on circles of radius $\sqrt{E_0}$ and $\sqrt{E_1}$ respectively as shown in the right-hand figure above. The distance between them is maximized if they are on opposite ends of the common diameter. In this case, $d_{01} = \sqrt{E_0} + \sqrt{E_1}$, $\cos\theta = -1$, and thus $P_{e,min} = Q\left(\frac{\sqrt{E_0}+\sqrt{E_1}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{E_0+E_1+2\sqrt{E_0E_1}}{2N_0}}\right)$

(f) Given that $s_0(t)$ is transmitted, the correlator outputs have joint pdf $(\pi N_0)^{-1}\exp[\{-(u-c_{00})^2-(v-c_{01})^2\}/N_0]$ The probability that **r** lies inside a circle of radius $d_{01}/2$ centered at $(c_{00},c_{01}) = \mathbf{s}_0$ is the integral of the pdf over the circle $= \int_{\rho=0}^{d_{01}/2}\int_{\varphi=0}^{2\pi} (\pi N_0)^{-1}\exp(-\rho^2/N_0)\bullet\rho d\varphi\bullet d\rho = \int_{\rho=0}^{d_{01}/2} 2(N_0)^{-1}\rho\bullet\exp(-\rho^2/N_0)\bullet d\rho = 1 - \exp(d_{01}{}^2/4N_0)$ upon making a change of variable as in Problem 2 of Problem Set #1.