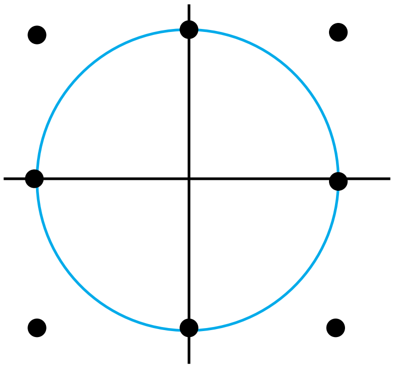
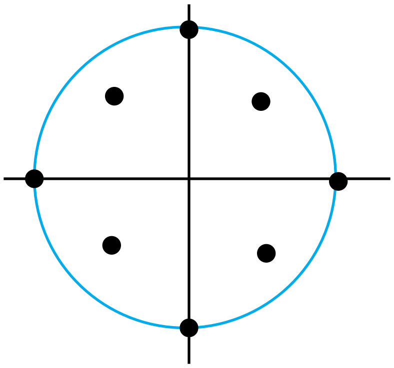
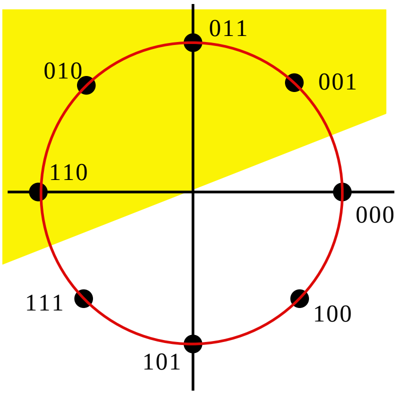
1.(a) The total energy is $4(r_0)^2 + 4(r_1)^2$. Since 3 bits are being transmitted, $E_b = (1/6)\cdot[(r_0)^2 + (r_1)^2]$.

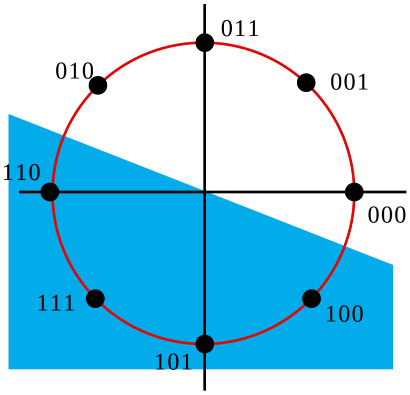
(b) Suppose the "outer" signal set is on a circle of radius 1, so that the signal coordinates are (±1,0), (0,±1). Then, the 8 signals form a QAM set if the other points are at (±1,±1) as shown on the left, and a rotated QAM set if they are at (±0.5,±0.5) as shown on the right. Thus, they lie on a circle of radius $\sqrt{2}$ or $\sqrt{1/2}$.

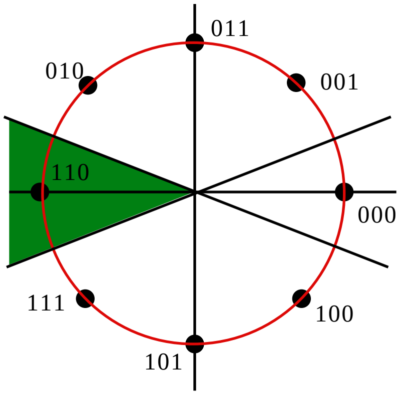




(c) Each signal is offset by π/4 along the circle from its two nearest neighbors. Hence, the distance between nearest neighbors is $2\cdot\sqrt{E}\cdot\sin(\pi/8)$. The probability of mistaking a signal for one nearest neighbor is thus $Q(2\cdot\sqrt{E}\cdot\sin(\pi/8)/\sqrt{2N_0})$ (cf. left-hand and middle figures below), and hence the nearest-neighbors approximation to the symbol error probability is $2\cdot Q(2\cdot\sqrt{E}\cdot\sin(\pi/8)/\sqrt{2N_0}) = 2\cdot Q\left(\sqrt{\frac{2E}{N_0}}\cdot\sin\left(\frac{\pi}{8}\right)\right)$







(d) The nearest neighbors approximations to the symbol error probability counts the probability of the shaded region in the rightmost figure above twice. Hence, the nearest neighbors approximation to the symbol error probability is an *upper bound on the exact value* of the symbol error probability.

(e) Given that **b** = 000 is transmitted, **or** that **b** = 001, **or** that **b** = 011 **or** that **b** = 010 is transmitted, the receiver output bit $\hat{b}_0 = 1$ if the observation **Z** lies in the shaded region in the left-hand figure on the next page. The boundary is at distance either $\sqrt{E}\cdot\sin(\pi/8)$ or $\sqrt{E}\cdot\sin(3\pi/8)$ from the signal point.

Consequently, $P\{\hat{b}_0 = 1 \mid \mathbf{b} = 000\} = P\{\hat{b}_0 = 1 \mid \mathbf{b} = 010\} = Q(\sqrt{2E/N_0}\cdot\sin(\pi/8))$

while $P\{\hat{b}_0 = 1 \mid \mathbf{b} = 001\} = P\{\hat{b}_0 = 1 \mid \mathbf{b} = 011\} = Q(\sqrt{2E/N_0}\cdot\sin(3\pi/8))$. Similar calculations (for conditional probabilities that $\hat{b}_0 = 0$) hold when 110, 111, 101, and 100 are transmitted. It follows that

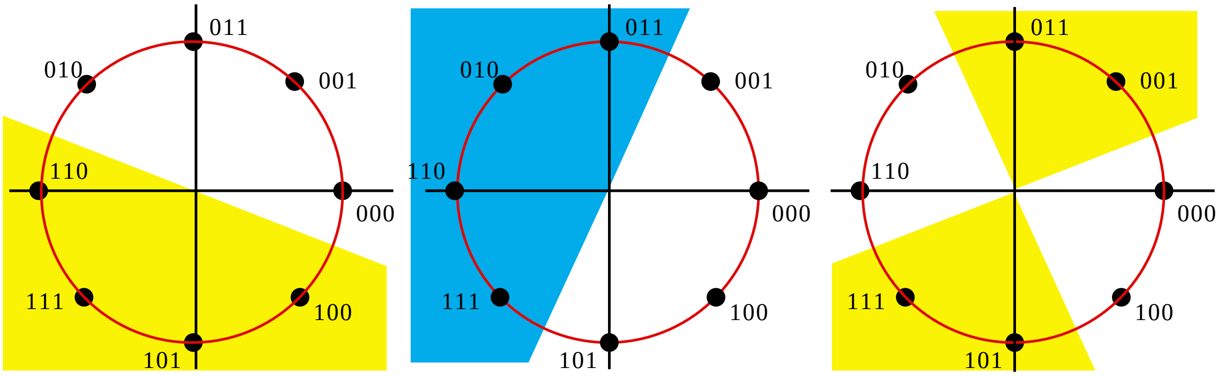
$$P\{\hat{b}_0 \neq b_0\} = \frac{1}{2}\cdot\left[Q\left(\sqrt{\frac{2E}{N_0}}\cdot\sin\left(\frac{\pi}{8}\right)\right) + Q\left(\sqrt{\frac{2E}{N_0}}\cdot\sin\left(\frac{3\pi}{8}\right)\right)\right]$$

Given that **b** = 000 is transmitted, **or** that **b** = 001, **or** that **b** = 100 **or** that **b** = 101 is transmitted, the receiver output bit $\hat{b}_1 = 1$ if the observation **Z** lies in the shaded region in the middle figure in the diagram on the next page. The boundary is at distance either $\sqrt{E}\cdot\sin(\pi/8)$ or $\sqrt{E}\cdot\sin(3\pi/8)$ from the signal point.

Consequently, $P\{\hat{b}_1 = 1 \mid \mathbf{b} = 001\} = P\{\hat{b}_1 =1 \mid \mathbf{b} = 101\} = Q(\sqrt{2E/N_0}\cdot\sin(\pi/8))$

while $P\{\hat{b}_1 = 1 \mid \mathbf{b} = 000\} = P\{\hat{b}_1 =1 \mid \mathbf{b} = 100\} = Q(\sqrt{2E/N_0}\cdot\sin(3\pi/8))$. Similar calculations (for conditional probabilities that $\hat{b}_1 = 0$)hold when 011, 010, 110, and 111 are transmitted. It follows that

$$P\{\hat{b}_1 \neq b_1\} = \frac{1}{2}\cdot\left[Q\left(\sqrt{\frac{2E}{N_0}}\cdot\sin\left(\frac{\pi}{8}\right)\right) + Q\left(\sqrt{\frac{2E}{N_0}}\cdot\sin\left(\frac{3\pi}{8}\right)\right)\right] = P\{\hat{b}_0 \neq b_0\}.$$



Given that **b** = 000 is transmitted, **or** that **b** = 001, **or** that **b** = 100 **or** that **b** = 101 is transmitted, the receiver output bit $\hat{b}_1 = 1$ if the observation **Z** lies in the shaded region in the middle figure above. The boundary is at distance either $\sqrt{E}\cdot\sin(\pi/8)$ or $\sqrt{E}\cdot\sin(3\pi/8)$ from the signal point.

Consequently, $P\{\hat{b}_1 = 1 \mid \mathbf{b} = 001\} = P\{\hat{b}_1 = 1 \mid \mathbf{b} = 101\} = Q(\sqrt{2E/N_0}\cdot\sin(\pi/8))$

while $P\{\hat{b}_1 = 1 \mid \mathbf{b} = 000\} = P\{\hat{b}_1 = 1 \mid \mathbf{b} = 100\} = Q(\sqrt{2E/N_0}\cdot\sin(3\pi/8))$. Similar calculations (for conditional probabilities that $\hat{b}_1 = 0$)hold when 011, 010, 110, and 111 are transmitted. It follows that

$$P\{\hat{b}_1 \neq b_1\} = \frac{1}{2}\cdot\left[Q\left(\sqrt{\frac{2E}{N_0}}\cdot\sin\left(\frac{\pi}{8}\right)\right) + Q\left(\sqrt{\frac{2E}{N_0}}\cdot\sin\left(\frac{3\pi}{8}\right)\right)\right] = P\{\hat{b}_0 \neq b_0\}.$$

Finally, from the right-hand diagram above and the previous analysis, we get that since
P(A B) = P(A) + P(B) – 2P(A)P(B) for independent events A and B,
$P\{\hat{b}_2 \neq 1 \mid \mathbf{b} = 000\} = P\{\hat{b}_2 = 1 \mid \mathbf{b} = 100\} = P\{\mathbf{Z}$ shaded region in right-hand diagram}

$$= Q\left(\sqrt{\frac{2E}{N_0}}\cdot\sin\left(\frac{\pi}{8}\right)\right) + Q\left(\sqrt{\frac{2E}{N_0}}\cdot\sin\left(\frac{3\pi}{8}\right)\right) - 2\cdot Q\left(\sqrt{\frac{2E}{N_0}}\cdot\sin\left(\frac{\pi}{8}\right)\right)\cdot Q\left(\sqrt{\frac{2E}{N_0}}\cdot\sin\left(\frac{3\pi}{8}\right)\right) = P\{\hat{b}_2 \neq b_2\}.$$

2(a) If a filter with impulse response $2p_T(t)\cos(2\pi(f_c+(4T)^{-1})t) = 2\cdot\mathrm{Re}\{p_T(t)\cdot\exp(j2\pi f_1 t)\}$ where $f_1 = f_c+(4T)^{-1}$ has as input the signal $A\cdot p_T(t)\cdot\cos(2\pi(f_c-(4T)^{-1})t) = \mathrm{Re}\{A\cdot p_T(t)\cdot\exp(-j\pi t/T)\cdot\exp(j2\pi f_1 t)\}$ then, with respect to the center frequency f_1, the complex baseband equivalent of the response of the filter is the convolution

$$p_T(t)*Ap_T(t)\exp\left(\frac{-j\pi t}{T}\right) = \begin{cases} \displaystyle\int_0^{\tau} A\cdot\exp\left(\frac{-j\pi t}{T}\right)dt, & 0 \le \tau \le T \\ \displaystyle\int_{\tau-T}^{T} A\cdot\exp\left(\frac{-j\pi t}{T}\right)dt, & T \le \tau \le 2T \end{cases} = \left(\frac{AT}{j\pi}\right)\left[1-\exp\left(\frac{-j\pi\tau}{T}\right)\right],\ 0\le\tau\le 2T.$$

Hence the output is an RF pulse given by $\mathrm{Re}\left\{\left(\frac{AT}{j\pi}\right)\cdot p_{2T}(t)\cdot\left[1 - \exp\left(\frac{-j\pi\tau}{T}\right)\right]\cdot\exp\left[j2\pi\left(f_c+\frac{1}{4T}\right)t\right]\right\}$

$$= \mathrm{Re}\left\{\left(\frac{2AT}{\pi}\right)\cdot p_{2T}(t)\cdot\left(\frac{1}{2j}\right)\cdot\left[\exp\left(\frac{j\pi t}{2T}\right) - \exp\left(\frac{-j\pi t}{2T}\right)\right]\cdot\exp(j2\pi f_c t)\right\}$$

$$= \mathrm{Re}\left\{\left(\frac{2AT}{\pi}\right)\cdot p_{2T}(t)\cdot\sin\left(\frac{\pi t}{2T}\right)\cdot\exp(j2\pi f_c t)\right\} = \left(\frac{2AT}{\pi}\right)\cdot p_{2T}(t)\cdot\sin\left(\frac{\pi t}{2T}\right)\cdot\cos(2\pi f_c t).$$

(b) The complex envelope of the input is now $A\cdot\left[(-1)^{a_1}\cdot p_T(t) + (-1)^{a_2}\cdot p_T(t-T)\right]$. By linearity and superposition, the complex envelope of the output signal is given by

$$(-1)^{a_1}\bullet\left(\frac{AT}{j\pi}\right)\bullet p_{2T}(t)\bullet\left[1-\exp\left(\frac{-j\pi t}{T}\right)\right]+(-1)^{a_2}\bullet\left(\frac{AT}{j\pi}\right)\bullet p_{2T}(t-T)\bullet\left[1-\exp\left(\frac{-j\pi(t-T)}{T}\right)\right].$$

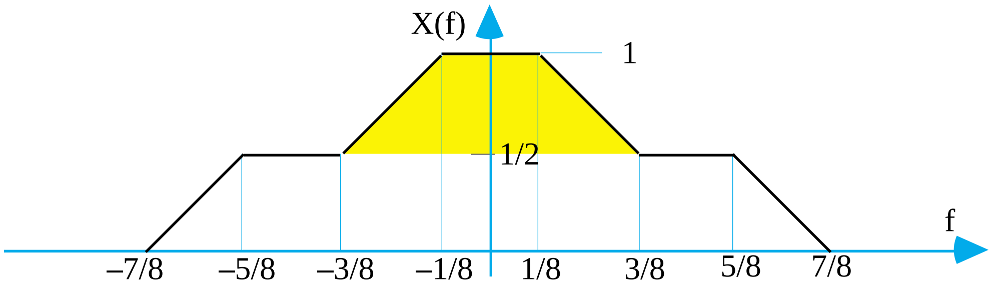
Since $p_{2T}(t) = p_{2T}(t-T) = 1$ for $T \le t < 2T$, and $\exp(j\pi) = -1$, then, during $[T,2T)$, the complex envelope is given by $\left(\frac{AT}{j\pi}\right)\bullet\left\{(-1)^{a_1}\bullet\left[1-\exp\left(\frac{-j\pi t}{T}\right)\right]+(-1)^{a_2}\bullet\left[1+\exp\left(\frac{-j\pi t}{T}\right)\right]\right\}$. Consequently, the filter output is

$$\mathrm{Re}\left\{\left(\frac{AT}{j\pi}\right)\bullet\left((-1)^{a_1}\bullet\left[1-\exp\left(\frac{-j\pi\tau}{T}\right)\right]+(-1)^{a_2}\bullet\left[1+\exp\left(\frac{-j\pi\tau}{T}\right)\right]\right)\bullet\exp\left[j2\pi\left(f_c+\frac{1}{4T}\right)t\right]\right\}$$

$$=(-1)^{a_1}\bullet\left(\frac{2AT}{\pi}\right)\bullet\sin\left(\frac{\pi t}{2T}\right)\bullet\cos(2\pi f_c t)+\mathrm{Re}\left\{\left(\frac{AT}{j\pi}\right)\bullet(-1)^{a_2}\bullet\left[1+\exp\left(\frac{-j\pi\tau}{T}\right)\right]\bullet\exp\left[j2\pi\left(f_c+\frac{1}{4T}\right)t\right]\right\}$$

$$=(-1)^{a_1}\bullet\left(\frac{2AT}{\pi}\right)\bullet\sin\left(\frac{\pi t}{2T}\right)\bullet\cos(2\pi f_c t)+(-1)^{a_2}\bullet\left(\frac{2AT}{\pi}\right)\bullet\cos\left(\frac{\pi t}{2T}\right)\bullet\sin(2\pi f_c t)$$ which is an MSK signal.

(c) If $a_1 = a_2$, then, during $[T,2T)$ the signal is $\pm(2AT/\pi)\bullet\sin(2\pi f_c t + \pi t/2T) = \pm(2AT/\pi)\bullet\sin(2\pi(f_c+1/4T)t)$ while if $a_1 \ne a_2$, then, during $[T,2T)$ the signal is $\pm(2AT/\pi)\bullet\sin(2\pi f_c t - \pi t/2T) = \pm(2AT/\pi)\bullet\sin(2\pi(f_c-1/4T)t)$. In short, the frequency is $f_c+1/4T$ if the bits are equal, and $f_c-1/4T$ if the bits are different.



3. To obtain the inverse Fourier transform of x(t), we note that X(f) is the sum of two trapezoidal functions as shown above. But each trapezoid is the convolution of two rect() functions of unequal width. In particular, the shaded traapezoid is $2\bullet\mathrm{rect}(4f)*\mathrm{rect}*(2f) = 2\bullet\mathrm{rect}(f/0.25)*\mathrm{rect}(f/0.5)$ while the unshaded trapezoid is $2\bullet\mathrm{rect}(4f)*\mathrm{rect}(2f/3) = 2\bullet\mathrm{rect}(f/0.25)*\mathrm{rect}(f/1.5)$. Hence,the convolution theorem and the table of Fourier transforms provided gives us that

$$\begin{aligned}x(t) &= 2\bullet[0.25\bullet\mathrm{sinc}(0.25t)\bullet0.5\bullet\mathrm{sinc}(0.5t) + 0.25\bullet\mathrm{sinc}(0.25t)\bullet1.5\bullet\mathrm{sinc}(1.5t)]\\ &= 0.5\bullet\mathrm{sinc}(0.25t)\bullet[0.5\bullet\mathrm{sinc}(0.5t) + 1.5\bullet\mathrm{sinc}(1.5t)]\\ &= 0.5\bullet\mathrm{sinc}(0.25t)\bullet(1/\pi t)\bullet[\sin(\pi t/2) + \sin(3\pi t/2)]\end{aligned}$$

But, $\sin(A+B) + \sin(A-B) = 2\bullet\sin(A)\bullet\cos(B)$ which gives (with $A = \pi t$, $B = \pi t/2$) that
$x(t) = 0.5\bullet\mathrm{sinc}(0.25t)\bullet(1/\pi t)\bullet2\bullet\sin(\pi t)\cos(\pi t/2) = \mathrm{sinc}(0.25t)\bullet\mathrm{sinc}(t)\bullet\cos(\pi t/2)$.
This is obviously a Nyquist pulse for signaling intervals of $T = 1$ and also $T = 4$. The latter value of T is implicit in the former for this special case, but not in the more general problem (cf. Blahut, Problem 2.13).

4.(a) The signals are orthogonal because the *baseband* signals $p_{T/2}(t)$ and $p_{T/2}(t-T/2)$ have non-overlapping support, i.e. $s_0(t)s_1(t) = 0$ for all t, and hence are orthogonal signals.

(b) Given that s_0 was transmitted and $\theta = \pi/4$, $\mathbf{U}_0$, $\mathbf{V}_0$, $\mathbf{U}_1$, and $\mathbf{V}_1$ are independent Gaussian random variables with common variance

$$\sigma^2 = \frac{N_0}{2}\int_0^{T/2}4\bullet\cos^2(2\pi f_c t)dt = \frac{N_0}{2}\int_0^{T/2}4\bullet\sin^2(2\pi f_c t)dt = \frac{N_0}{2}\int_{T/2}^{T}4\bullet\cos^2(2\pi f_c t)dt = \frac{N_0}{2}\int_{T/2}^{T}4\bullet\sin^2(2\pi f_c t)dt = \frac{N_0T}{2}.$$

$$E[\mathbf{U}_0] = \int_0^{T/2}\sqrt{2}A\bullet\cos(2\pi f_c t+\pi/4)\bullet2\bullet\cos(2\pi f_c t)dt = \sqrt{2}A\bullet(T/2)\bullet\cos(\pi/4),\ E[\mathbf{U}_1] = 0 \text{ (no signal in (T/2,T)!!}$$

$$E[\mathbf{V}_0] = \int_0^{T/2}\sqrt{2}A\bullet\cos(2\pi f_c t+\pi/4)\bullet(-2)\bullet\sin(2\pi f_c t)dt = \sqrt{2}A\bullet(T/2)\bullet\sin(\pi/4),\ E[\mathbf{V}_1] = 0 \text{ (no signal in (T/2,T)!!}$$

(c) This is noncoherent demodulation of orthogonal signals, and the error probability is $(1/2)\bullet\exp(-(A_0)^2/4\sigma^2)$ where $(A_0)^2 = (E[\mathbf{U}_0])^2 + (E[\mathbf{V}_0])^2 = 2A^2(T/2)^2 = A^2T^2/2$. Hence, $P_e = (1/2)\bullet\exp(-A^2T/4N_0) = (1/2)\bullet\exp(-E/2N_0)$ since the energy in each pulse of rms amplitude A is $E = A^2(T/2)$.