University of Illinois	**Final Exam** **Friday, May 11, 2001** **8:00 a.m — 11:00 a.m.**	**ECE 361** **Spring 2001**

Instructions: **200 points**

Two pages of notes allowed. No other notes, books, tables of integrals, or calculators/personal computers permitted.

Notation: Gaussian density function: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$ $-\infty < x < \infty$.

$\Phi(x)$ = cumulative probability distribution function for standard Gaussian random variable

$Q(x) = 1 - \Phi(x)$

AWGN denotes additive (zero-mean) white Gaussian noise with power spectral density $N_0/2$. This random process is independent of the choice of transmitted signal.

$P_{e,i}$ = probability of error given that signal $s_i(t)$ was transmitted

Rayleigh density function: $f(r) = \frac{r}{\sigma^2} \cdot \exp\left(-\frac{r^2}{2\sigma^2}\right)$, $r > 0$.

Rician density function: $f(r) = \frac{r}{\sigma^2} \cdot I_0\left(\frac{rA}{\sigma^2}\right) \cdot \exp\left(-\frac{r^2 - A^2}{2\sigma^2}\right)$, $r > 0$, where $I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp(x \cdot \cos\theta)\, d\theta$

Some more-or-less useless facts:

$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ $\quad$ $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

$2\cos^2(A) = 1 + \cos(2A)$ $\quad$ $2\sin^2(A) = 1 - \cos(2A)$

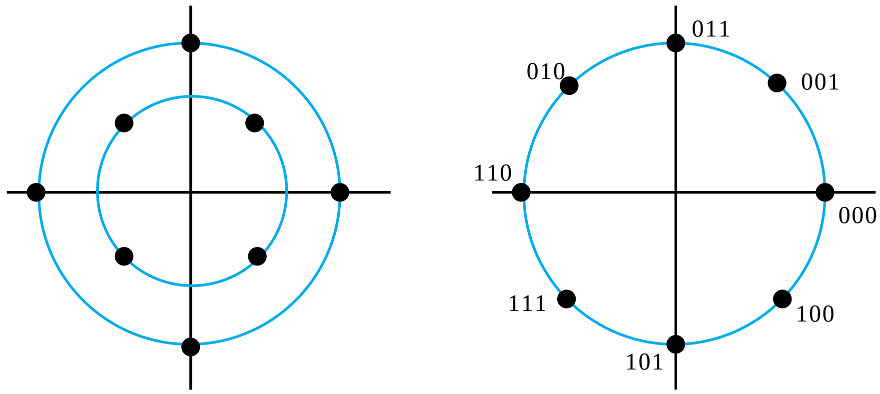
If $f_cT >> 1$, then $\int_a^{a+T} \cos^2(2\pi f_c t)dt \approx \int_a^{a+T} \sin^2(2\pi f_c t)dt \approx T/2$ and $\int_a^{a+T} \cos(2\pi f_c t)\sin(2\pi f_c t)dt \approx 0$, with exact equality holding in both cases if f_cT is an integer.

$$\text{rect}(t) = \begin{cases} 1 & \text{if } -1/2 \le t \le 1/2, \\ 0, & \text{otherwise.} \end{cases} \qquad p_T(t) = \begin{cases} 1, & \text{if } 0 \le t < T, \\ 0, & \text{otherwise.} \end{cases}$$

$X(t)$	$\leftrightarrow$	$x(-f)$	$x(\alpha t)$	$\leftrightarrow$	$\lvert\alpha\rvert^{-1} \cdot X(\alpha^{-1} f)$
$x^*(t)$	$\leftrightarrow$	$X^*(-f)$	$x^*(-t)$	$\leftrightarrow$	$X^*(f)$
$x(t-t_0)$	$\leftrightarrow$	$X(f) \cdot \exp(-j2\pi f t_0)$	$x(t) \cdot \exp(j2\pi f_0 t)$	$\leftrightarrow$	$X(f-f_0)$
$x(t) \cdot \cos(2\pi f_0 t)$	$\leftrightarrow$	$(1/2)[X(f+f_0) + X(f-f_0)]$	$x(t) \cdot \sin(2\pi f_0 t)$	$\leftrightarrow$	$(j/2)[X(f+f_0) - X(f-f_0)]$
$\frac{d}{dt} x(t)$	$\leftrightarrow$	$(j2\pi f) \cdot X(f)$	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\leftrightarrow$	$(j2\pi f)^{-1} \cdot X(f) + X(0) \cdot \delta(f)/2$
$x * y = \int_{-\infty}^{\infty} x(\tau) \cdot y(t-\tau) d\tau$	$\leftrightarrow$	$X(f) \cdot Y(f)$	$x(t) \cdot y(t)$	$\leftrightarrow$	$\int_{-\infty}^{\infty} X(\lambda) \cdot Y(f-\lambda) d\lambda$
$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$=$	$\int_{-\infty}^{\infty} \lvert X(f)\rvert^2 df$	$\int_{-\infty}^{\infty} x(t) y^*(t) dt$	$=$	$\int_{-\infty}^{\infty} X(f) Y^*(f) df$
$R_{X,y}(\tau)$	$=$	$\int_{-\infty}^{\infty} x(t+\tau) \cdot y^*(t)\, dt$	$R_{X,y}(\tau)$	$\leftrightarrow$	$S_{X,y}(f) = X(f)Y^*(f)$
$\text{rect}(t/T)$	$\leftrightarrow$	$T \cdot \text{sinc}(fT)$	$W \cdot \text{sinc}(Wt)$	$\leftrightarrow$	$\text{rect}(f/W)$

where $\text{sinc}(z) = \frac{\sin \pi z}{\pi z}$ is the *sinc function*

1. **(60 points)** A 2-dimensional 8-ary signal constellation is shown in the left-hand diagram below where 4 signals are a QPSK signal set lying on a circle of radius r_0 while the other 4 signals are a QPSK signal set lying on a circle of radius r_1. The QPSK sets are rotated by $\pi/4$ with respect to one another. The signals are equally likely to be transmitted.



(a) Express E_b, the **energy per bit,** of this signal set in terms of r_0 and r_1.

(b) Now, suppose that $r_0 = 1$ is fixed. Then, for two values of r_1, the signal constellation forms a QAM signal set (or a QAM signal set rotated by $\pi/4$). What are these values of r_1? (Hint: one value of r_1 is smaller than 1 and the other is larger than 1)

(c) Next, suppose that $r_0 = r_1 = \sqrt{E}$, so that the signal constellation is 8-ary PSK as illustrated in the diagram on the right above. On an AWGN channel with power spectral density $N_0/2$, what is the **nearest-neighbors approximation** to the symbol error probability?

(d) State clearly **which** of the four statements (i)-(iv) below is true:
the answer that you gave in part (c) is

- **(i)** the *exact value* of the symbol error probability
- **(ii)** an *upper bound* on the exact value of the symbol error probability
- **(iii)** a *lower bound* on the exact value of the symbol error probability
- **(iv)** neither an upper bound nor a lower bound, but *just an approximation* to the exact value of the symbol error probability

(e) Continue to suppose that $r_0 = r_1 = \sqrt{E}$, so that the signal constellation is 8-ary PSK operating on an AWGN channel with power spectral density $N_0/2$. Assume that 3 data bits $\underline{\mathbf{b}} = (b_0 b_1 b_2)$ are mapped to the 8-PSK signal set using a Gray code as shown in the right-hand diagram above, and let $\hat{\underline{\mathbf{b}}} = (\hat{b}_0 \hat{b}_1 \hat{b}_2)$ denote the receiver output.

Find $P\{\hat{b}_0 \neq b_0 \mid \underline{\mathbf{b}} = 000\}$ and $P\{\hat{b}_0 \neq b_0 \mid \underline{\mathbf{b}} = 001\}$. Use these results to find $P\{\hat{b}_0 \neq b_0\}$.

Find $P\{\hat{b}_1 \neq b_1 \mid \underline{\mathbf{b}} = 000\}$ and $P\{\hat{b}_1 \neq b_1 \mid \underline{\mathbf{b}} = 001\}$. Use these results to find $P\{\hat{b}_1 \neq b_1\}$.

Find $P\{\hat{b}_2 \neq b_2 \mid \underline{\mathbf{b}} = 000\}$ and $P\{\hat{b}_2 \neq b_2 \mid \underline{\mathbf{b}} = 100\}$. Use these results to find $P\{\hat{b}_2 \neq b_2\}$.

2. **(50 points)** An MSK signal consists of RF pulses of duration T at frequency $f_c + 1/(4T)$ or $f_c - 1/(4T)$ (depending on the data bits being transmitted). Assume that f_cT is an integer.

(a) Consider the RF pulse $A \cdot p_T(t) \cdot \cos(2\pi[f_c - 1/(4T)]t)$ passed through a narrowband filter with impulse response $h(t) = 2 \cdot p_T(t) \cdot \cos(2\pi[f_c + 1/(4T)]t)$. Calculate the complex envelope of the filter output with respect to a "center frequency" or "carrier frequency" of $f_1 = f_c + 1/(4T)$, and show that the output pulse can also be expressed as

$$(2AT/\pi) \cdot p_{2T}(t) \cdot \sin(\pi t/2T) \cdot \cos(2\pi f_c t).$$

Remember to use the narrowband approximations (and the trignometric identities on the cover page of the exam!) in your calculations…

(b) Now suppose that **two** successive RF pulses of the type considered in part (a) are passed through the filter. The output pulse thus has duration 3T.

Use linearity and superposition together with the result claimed in part (a) viz. the filter output is $(2AT/\pi) \cdot p_{2T}(t) \cdot \sin(\pi t/2T) \cdot \cos(2\pi f_c t)$ to deduce the output of the filter when the input is

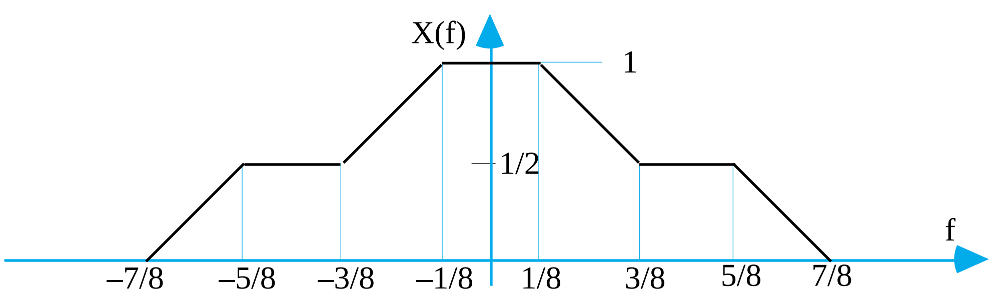
$$A \cdot \left[(-1)^{a_1} \cdot p_T(t) + (-1)^{a_2} \cdot p_T(t-T)\right] \cdot \cos(2\pi[f_c - 1/(4T)]t).$$

You need to specify the output during the time interval [T,2T) only.

(c) How is the frequency of the filter output signal during the time interval [T,2T) related to the data bits a_1 and a_2?

Note that the filter input is just a standard binary PSK signal at carrier frequency $f_c - 1/(4T)$ but the output (during [T,2T) is an MSK signal. This method of filtering a PSK signal at carrier frequency $f_c - 1/(4T)$ through a narrowband filter with center frequency $f_c + 1/(4T)$ provides an alternative way of generating an MSK signal.

3. **(40 points)** A signal x(t) has Fourier transform as shown. Find **all** signaling intervals T for which x(t) is a Nyquist pulse. Hint: the convolution of two rect()s is a trapezoid.



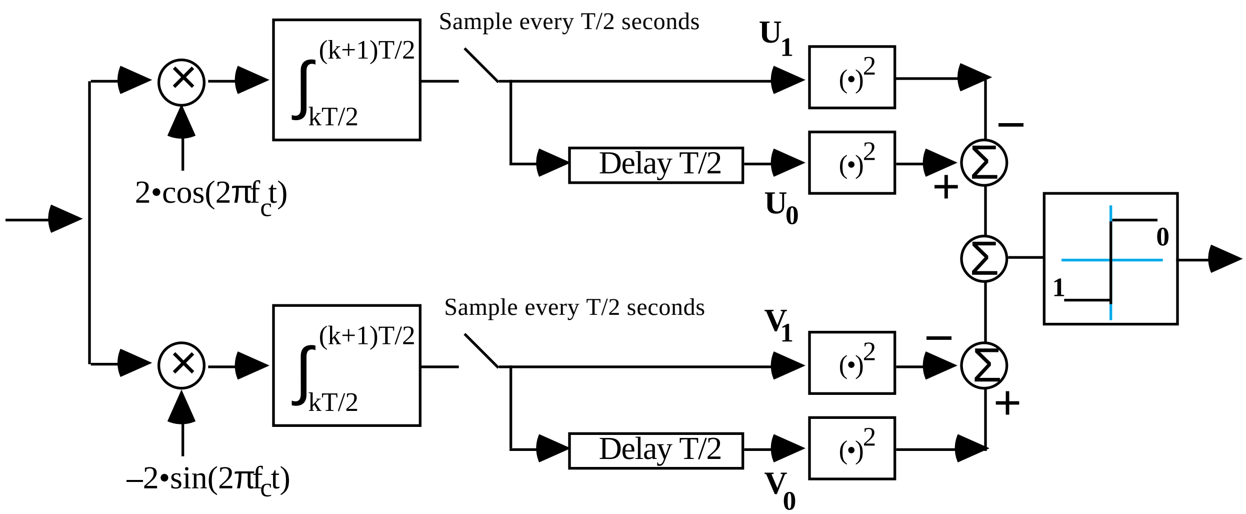
4. **(50 points)** A PPM system operating on an AWGN channel uses signals

$$s_0(t) = \sqrt{2}A \cdot p_{T/2}(t) \cdot \cos(2\pi f_c t + \theta)$$

$$s_1(t) = \sqrt{2}A \cdot p_{T/2}(t - T/2) \cdot \cos(2\pi f_c t + \theta)$$

where $f_c T$ is a large even integer and θ is unknown.

A noncoherent correlation receiver for this signal set is shown below where the integrator outputs are sampled at integer multiples of T/2. The diagram indicates the state of affairs at time T. **Note that the integrator is dumped (that is, its output is reset to 0 and the integration is re-started) at the end of each T/2 second period. The decision is made once every T seconds.**



As shown, $\mathbf{U}_0$ and $\mathbf{V}_0$ denote the sample values at time t = T/2 and $\mathbf{U}_1$ and $\mathbf{V}_1$ denote the sample values at time t = T. The receiver computes $\{(\mathbf{U}_0)^2 + (\mathbf{V}_0)^2\} - \{(\mathbf{U}_1)^2 + (\mathbf{V}_1)^2\}$ and decides that $s_0(t)$ was transmitted if this quantity is positive.

(a) Explain why $s_0(t)$ and $s_1(t)$ are orthogonal over the interval [0,T] even though they are at the same carrier frequency f_c and have the same RF phase θ.

(b) Compute the means and variances of $\mathbf{U}_0$, $\mathbf{V}_0$, $\mathbf{U}_1$, and $\mathbf{V}_1$ conditioned on $s_0(t)$ being transmitted and $\theta = \pi/4$.

(c) **Use** the results of part (b) to **state** the error probability of this receiver in terms of one or more of the parameters A, T, f_c, θ and N_0. You are **not asked** to provide a derivation of your answer via integration of pdfs etc.