University of Illinois Spring 2004

ECE 361: Solutions to First MidSemester Exam

1. **(a)** The signal outputs from the receiver are easily found to be

$$\hat{s}_i(T) = \left. s_i(t) \star h(t) \right|_{t=T} = \begin{cases} A^2T & \text{if } i = 0, \\ A^2T/2 & \text{if } i = 1, \end{cases}$$

while the noise output has variance $\sigma^2 = \int_0^T h^2(t)(N_0/2)dt = \int_0^T A^2(N_0/2)dt = A^2TN_0/2$. Hence, the minimax error probability is

$$P_e = Q\left(\frac{\hat{s}_0(T) - \hat{s}_1(T)}{2\sigma}\right) = Q\left(\sqrt{\frac{A^2T}{8N_0}}\right).$$

(b) Now $\hat{s}_0(T) = A^2T/2$ while $\hat{s}_1(T) = \int_0^T A^2(t/T)(T-t)/Tdt = A^2T/6$. The noise output has variance $\sigma^2 = \int_0^T h^2(t)(N_0/2)dt = \int_0^T A^2(t/T)^2(N_0/2)dt = A^2TN_0/6$. Hence, the minimax error probability is

$$P_e = Q\left(\frac{\hat{s}_0(T) - \hat{s}_1(T)}{2\sigma}\right) = Q\left(\sqrt{\frac{A^2T}{6N_0}}\right).$$

(c) The signals are of energies A^2T and $A^2T/3$ respectively while their inner product is $\langle s_0, s_1\rangle = A^2T/2$ (cf. part (a)). Hence, the optimum minimax receiver has error probability

$$P_e = Q\left(\sqrt{\frac{E_0 + E_1 - 2\langle s_0, s_1\rangle}{2N_0}}\right) = Q\left(\sqrt{\frac{A^2T}{6N_0}}\right).$$

Since $Q(\cdot)$ is a decreasing function of its argument, the answer in part (c) is smaller than the answer in part (a). The answer in part (c) is *the same* as the answer in part (b) because the filter in part (b) is, in fact, the matched filter

$$h_m(t) = s_0(T-t) - s_1(T-t) = A - A((T-t)/T) = At/T = s_1(t) \text{ for } 0 \le t \le T.$$

I had meant to change the filters in parts (a) and (b) to $s_0(T-t)$ and $s_1(T-t)$ (which makes no difference in part (a) but does change part (b) considerably, but forgot ....

2. **(a)** The optimum minimax error probability is achieved by putting the signals as far apart as possible in signal space. Since the signals are at distances $\sqrt{E_0}$ and $\sqrt{E_1}$ from the origin, we can set them as far apart as $d = \sqrt{E_0} + \sqrt{E_1}$ by choosing $s_1(t) = -\sqrt{E_1/E_0}s_0(t)$ and thus achieving minimax error probability

$$Q\left(\frac{d}{\sqrt{2N_0}}\right) = Q\left(\frac{\sqrt{E_0} + \sqrt{E_1}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{E_0 + E_1 + 2\sqrt{E_0E_1}}{2N_0}}\right)$$

where the last equality follows from the fact that $\langle s_0, s_1\rangle = -\sqrt{E_0E_1}$. Note also that the Cauchy-Schwarz Inequality says that $\sqrt{E_0E_1}$ is the minimum value of the inner product and is achieved when one signal is a scalar multiple of the other.

(b) Once again, the general principle is that the signals must be placed as far apart as possible. Given the energy constraints, $s_0(t)$ and $s_2(t)$ should be chosen to be antipodal (and similarly $s_1(t)$ and $s_3(t)$ should be chosen to be antipodal) so as to maximize the distance between them. Assuming without loss of generality that $\mathbf{s}_0$ and $\mathbf{s}_2$ have coordinates $(\pm\sqrt{E_0}, 0)$ in the signal space, then choosing $\mathbf{s}_1$ and $\mathbf{s}_3$ to be $(0, \pm\sqrt{E_1})$ makes the minimum distance between the even-subscripted and odd-subscripted signals achieve its maximum value $\sqrt{E_0 + E_1}$. More formally, maximizing the distance is the same as minimizing the both inner products $\langle \mathbf{s}_0, \mathbf{s}_1\rangle$ and $\langle \mathbf{s}_0, \mathbf{s}_3\rangle = -\langle \mathbf{s}_0, \mathbf{s}_1\rangle$, and so it is best to make $\langle \mathbf{s}_0, \mathbf{s}_1\rangle = 0$ by choosing $\mathbf{s}_1$ and $\mathbf{s}_3$ to be $(0, \pm\sqrt{E_1})$.

(c) It is easily seen that $d_{0,1} = d_{0,3} = \sqrt{E_0 + E_1}$ while $d_{0,2} = 2\sqrt{E_0}$ and $d_{1,3} = 2\sqrt{E_1}$. To reduce the bit error probability, we should make double bit errors unlikely by assigning complementary bit pairs to signals as far away as possible. If $E_0 < 3E_1$, then $\sqrt{E_0 + E_1} < 2\sqrt{E_1} < 2\sqrt{E_0}$ and Gray coding works:

$$00 \leftrightarrow \mathbf{s}_0, \quad 01 \leftrightarrow \mathbf{s}_1, \quad 11 \leftrightarrow \mathbf{s}_2, \quad 10 \leftrightarrow \mathbf{s}_3,$$

since the smallest distances correspond to single bit errors.
On the other hand, if $E_0 > 3E_1$, then $2\sqrt{E_1} < \sqrt{E_0 + E_1} < 2\sqrt{E_0}$ and in this case natural binary coding is better:

$$00 \leftrightarrow \mathbf{s}_0, \quad 01 \leftrightarrow \mathbf{s}_1, \quad 10 \leftrightarrow \mathbf{s}_2, \quad 11 \leftrightarrow \mathbf{s}_3,$$

since once again the smallest distances correspond to single bit errors. Notice that Gray coding would give large separation to 00 and 11 but put 01 and 10 too close together.

(d) The 4-QAM (or QPSK) signal set $\{(\pm\sqrt{E/2}, \pm\sqrt{E/2})\}$ (or any rotation thereof, such as, for example: $\{(\pm\sqrt{E}, 0), (0, \pm\sqrt{E})\}$) is the optimal placement of 4 equal-energy signals in two-dimensional signal space. Thus, let $\{\mathbf{s}_1, \mathbf{s}_3, \mathbf{s}_5, \mathbf{s}_7\} = \{(\pm\sqrt{E_1}, 0), (0, \pm\sqrt{E_1})\}$. On the other hand, choosing $\{\mathbf{s}_0, \mathbf{s}_2, \mathbf{s}_4, \mathbf{s}_6\} = \{(\pm\sqrt{E_0}, 0), (0, \pm\sqrt{E_0})\}$ puts signals from one set at distance only $\sqrt{E_0} - \sqrt{E_1} = \sqrt{E_0 + E_1 - 2(E_0E_1)^{1/2}}$ from signals in the other set. A little thought (and drawing) shows that it is best to choose $\{\mathbf{s}_0, \mathbf{s}_2, \mathbf{s}_4, \mathbf{s}_6\} = \{(\pm\sqrt{E_0/2}, \pm\sqrt{E_0/2})\}$ thereby achieving a slightly larger separation of $\sqrt{E_0 + E_1 - (2E_0E_1)^{1/2}}$.

(e) If $E_0 = 2E_1$, then the signals are placed at $\{(\pm\sqrt{E_1}, 0), (0, \pm\sqrt{E_1})\}, \{(\pm\sqrt{E_1}, \pm\sqrt{E_1})\}$, i.e. at the vertices and the midpoints of the sides of a square of side $2\sqrt{E_1}$ centered at the origin. It is easy to see that $d_{\min} = \sqrt{E_1}$.

3. **(a)** The average energy per symbol is $\dfrac{(2^{2m}-1)d^2}{6}$ and the average energy per bit is $\dfrac{(2^{2m}-1)d^2}{12m}$.

(b) Let α denote $d/\sqrt{2N_0}$. Then, the average symbol error probability is

$$4(1 - 2^{-m})Q(\alpha) - 4(1 - 2^{-m})^2 Q^2(\alpha).$$

(c) $(\mathcal{X}_0, \mathcal{X}_1)$ are independent Gaussian random variables with mean point $\mathbf{s}_j$ and common variance $N_0/2$. The probability that $(\mathcal{X}_0, \mathcal{X}_1)$ lies within distance $d/2$ from $\mathbf{s}_j$ is thus the integral of the joint pdf over the circle of radius $d/2$ centered at $\mathbf{s}_j$, which is given by

$$\int_{r=0}^{d/2} \int_{\theta=0}^{2\pi} \frac{1}{\pi N_0} \exp\left(\frac{-r^2}{N_0}\right) r\,d\theta\,dr = \int_{r=0}^{d/2} \frac{2r}{N_0} \exp\left(\frac{-r^2}{N_0}\right) dr = -\exp\left(\frac{-r^2}{N_0}\right)\Bigg|_0^{d/2} = 1 - \exp\left(\frac{-\alpha^2}{2}\right)$$

$$= 1 - \exp(-d^2/4N_0).$$

The probability that the optimum receiver makes a correct decision is $P\{(\mathcal{X}_0, \mathcal{X}_1) \in \Gamma_j | \mathbf{s}_j\}$ which has value $1 - 2Q(\alpha) + Q^2(\alpha) = (1 - Q(\alpha))^2$, or $1 - 3Q(\alpha) + 2Q^2(\alpha) = (1 - Q(\alpha))(1 - 2Q(\alpha)$, or $1 - 4Q(\alpha) + 4Q^2(\alpha) = (1 - 2Q(\alpha))^2$ for corner, edge, and interior points respectively. This last probability (which is the smallest of the three probabilities above) is the integral of the joint pdf over a square of side d centered at $\mathbf{s}_j$, and is thus larger than $1 - \exp(-\alpha^2/2)$, the integral over the circle of radius $d/2$. Since the average probability of a correct decision is a weighted sum of the above three probabilities, this average probability is also larger than $1 - \exp(-\alpha^2/2)$.

In general, bounded-distance modulation results in a smaller probability of correct decision *but leads to a very significant reduction in the probability of an incorrect decision.* In many applications, a very small error probability is much to be desired even if it means requests for re-transmission (and thus a smaller throughput).