

## ECE 361: Solutions to Second MidSemester Exam

1. Parts (a)-(c) can be solved most easily by noting that the *preference* regions  $\Gamma_{-1,+1}$ ,  $\Gamma_{+j,+1}$ , and  $\Gamma_{-j,+1}$  have the conditional probabilities stated. Since the preference regions are subsets of the *decision* regions, the result follows. More directly, we have

- (a) The receiver decides that  $s_{-1}$  was transmitted if and only if the event

$$\{|\mathcal{Z}_1| > \max\{|\mathcal{Z}_2|, |\mathcal{Z}_3|, \dots, |\mathcal{Z}_{M/2}|\}\} \cap \{\mathcal{Z}_1 < 0\}$$

occurs. This event is a *subset* of the event  $\{\mathcal{Z}_1 < 0\}$  which has probability  $Q(\sqrt{2E_s/N_0})$  since  $\mathcal{Z}_1 \sim \mathcal{N}(\sqrt{E_s}, N_0/2)$ .

- (b) The receiver decides that  $s_{+j}$  was transmitted if and only if the event

$$\{|\mathcal{Z}_j| > \max\{|\mathcal{Z}_1|, \dots, |\mathcal{Z}_{j-1}|, |\mathcal{Z}_{j+1}|, \dots, |\mathcal{Z}_{M/2}|\}\} \cap \{\mathcal{Z}_j > 0\}$$

occurs. This event is a *subset* of the event  $\{\mathcal{Z}_j > \mathcal{Z}_1\}$  which has probability  $Q(\sqrt{E_s/N_0})$  since  $\mathcal{Z}_j - \mathcal{Z}_1 \sim \mathcal{N}(-\sqrt{E_s}, N_0)$ .

- (c) Similarly, the probability of the event

$$\{|\mathcal{Z}_j| > \max\{|\mathcal{Z}_1|, \dots, |\mathcal{Z}_{j-1}|, |\mathcal{Z}_{j+1}|, \dots, |\mathcal{Z}_{M/2}|\}\} \cap \{\mathcal{Z}_j < 0\}$$

is bounded above  $P\{-\mathcal{Z}_j > \mathcal{Z}_1\} = P\{\mathcal{Z}_j + \mathcal{Z}_1 < 0\} = Q(\sqrt{E_s/N_0})$  since  $\mathcal{Z}_j + \mathcal{Z}_1 \sim \mathcal{N}(\sqrt{E_s}, N_0)$ .

- (d) Under the given conditions, the decision statistic with the largest absolute value is either  $\mathcal{Z}_1$  or  $\mathcal{Z}_2$ . Hence,  $s_{\pm j}$ ,  $j > 2$  will never be chosen by the receiver. Furthermore, if  $|\mathcal{Z}_1|$  is the largest, then  $s_{+2}$  will not be chosen; if  $|\mathcal{Z}_2|$  is the largest,  $s_{+2}$  still will not be chosen since  $\mathcal{Z}_2 < 0$  by assumption. Thus,  $s_{+2}$  also will never be chosen by the receiver under the given conditions.

- (e)  $s_{-2}$  will be chosen if  $|\mathcal{Z}_1| < x\sqrt{N_0/2}$ . Since  $\mathcal{Z}_1 \sim \mathcal{N}(\sqrt{E_s}, N_0)$ , we get the desired probability to be  $Q(\mu - x) - Q(\mu + x)$  where  $\mu = \sqrt{2E_s/N_0}$ .

- (f) The conditional probability that the receiver chooses  $s_{+2}$  is 0 as shown already in part (d).

- (g) The conditional probability that the receiver decides that  $s_{-1}$  was transmitted is

$$P\{\mathcal{Z}_1 < -x\sqrt{N_0/2}\} = Q(\mu + x).$$

- (h) The conditional probability of error under the given conditions is the sum of the answers to parts (e)-(g) which is  $Q(\mu - x)$ .

Note that the probability that  $\mathcal{Z}_2$  is the best of the rest is  $[1 - 2Q(x)]^{M/2-2}$  since all the other  $\mathcal{Z}_i$ 's must be in the range  $(-x\sqrt{N_0/2}, +x\sqrt{N_0/2})$ . Hence, the probability of error when  $\mathcal{Z}_\epsilon < 0$  is obtained by multiplying  $Q(\mu - x)$  by  $\phi(x)$ , the standard Gaussian pdf (why??), and integrating from  $x = 0$  to  $x = \infty$ . On the other hand, if  $\mathcal{Z}_\epsilon = +x\sqrt{N_0/2} > 0$ , is the best of the rest, then the receiver makes an error if  $\mathcal{Z}_1 < x\sqrt{N_0/2}$  which has probability  $Q(\mu - x)$  while the probability of  $\mathcal{Z}_\epsilon$  being the best of the rest is still  $[1 - 2Q(x)]^{M/2-2}$ . Hence, the probability of error conditioned on  $\mathcal{Z}_\epsilon$  being the best of the rest is

$$2 \int_0^\infty [1 - 2Q(x)]^{M/2-2} Q(\mu - x) \phi(x) dx.$$

But, since *any* of the  $M/2 - 1$   $\mathcal{Z}_i$ 's are equally likely to be the best of the rest, we get the error probability for  $M$ -ary biorthogonal signaling as

$$\int_0^\infty (M - 2)[1 - 2Q(x)]^{M/2-2} Q(\mu - x) \phi(x) dx$$

as stated on page 173 of my Lecture Notes on the class web page.

2. (a) Since both  $s$  and  $h$  have support  $(-T/2, T/2)$  and are even functions of  $t$ , the convolution  $s \star h$  is an even function with support  $(-T, T)$ . Ignoring double frequency terms, we get that for  $0 \leq \tau \leq T$ ,

$$\begin{aligned}
(s \star h)(\tau) &= \int_{-T/2+\tau}^{T/2} 2 \cdot \cos\left(2\pi\left(f_c + \frac{1}{4T}\right)t\right) \cos\left(2\pi\left(f_c - \frac{1}{4T}\right)(\tau - t)\right) dt \\
&= \int_{-T/2+\tau}^{T/2} \cos\left(\frac{\pi t}{T} + 2\pi f_c \tau - \frac{\pi \tau}{2T}\right) dt \\
&= \left(\frac{T}{\pi}\right) \sin\left(\frac{\pi t}{T} + 2\pi f_c \tau - \frac{\pi \tau}{2T}\right) \Big|_{-T/2+\tau}^{T/2} \\
&= \left(\frac{T}{\pi}\right) \left[ \sin\left(\frac{\pi}{2} + 2\pi f_c \tau - \frac{\pi \tau}{2T}\right) - \sin\left(\frac{\pi(-T/2 + \tau)}{T} + 2\pi f_c \tau - \frac{\pi \tau}{2T}\right) \right] \\
&= \left(\frac{T}{\pi}\right) \left[ \sin\left(\frac{\pi}{2} + 2\pi f_c \tau - \frac{\pi \tau}{2T}\right) - \sin\left(\frac{-\pi}{2} + 2\pi f_c \tau + \frac{\pi \tau}{2T}\right) \right] \\
&= \left(\frac{T}{\pi}\right) \left[ \cos\left(2\pi f_c \tau - \frac{\pi \tau}{2T}\right) + \cos\left(2\pi f_c \tau + \frac{\pi \tau}{2T}\right) \right]
\end{aligned}$$

since  $\sin(\pm\pi/2 + x) = \pm \cos(x)$ . But,  $\cos(x - y) + \cos(x + y) = 2 \cos(x) \cos(y)$  giving that

$$(s \star h)(\tau) = \left(\frac{2T}{\pi}\right) \cdot \text{rect}\left(\frac{\tau}{2T}\right) \cos\left(\frac{\pi \tau}{2T}\right) \cos(2\pi f_c \tau).$$

- (b) Delaying  $\cos(2\pi(f_c - (4T)^{-1})t)$  by  $T$  gives  $\cos(2\pi(f_c - 1/4T)t + \pi/2)$ , not  $\cos(2\pi(f_c - 1/4T)t)$  as needed in the problem, whereas delaying  $\cos(2\pi(f_c - (4T)^{-1})t - \pi/2) = \sin(2\pi(f_c - (4T)^{-1})t)$  by  $T$  gives the desired signal  $\cos(2\pi(f_c - 1/4T)t)$ . From the derivation in part (a), we see that the filter response to  $\text{rect}(t/T) \cos(2\pi(f_c - (4T)^{-1})t - \pi/2)$  can be found by carrying along a  $\pi/2$  term, which merely changes the  $\cos(2\pi f_c \tau)$  above to  $\sin(2\pi f_c \tau)$ . Thus, the output signal over  $(0, T)$  is

$$\begin{aligned}
(s \star h)(\tau) &= \left(\frac{2T}{\pi}\right) \left[ (-1)^{b_0} \cos\left(\frac{\pi \tau}{2T}\right) \cos(2\pi f_c \tau) + (-1)^{b_1} \cos\left(\frac{\pi(\tau - T)}{2T}\right) \sin(2\pi f_c(\tau - T)) \right] \\
&= \left(\frac{2T}{\pi}\right) \left[ (-1)^{b_0} \cos\left(\frac{\pi \tau}{2T}\right) \cos(2\pi f_c \tau) + (-1)^{b_1} \sin\left(\frac{\pi \tau}{2T}\right) \sin(2\pi f_c \tau) \right]
\end{aligned}$$

since  $f_c T$  is an integer. Ignoring the trivial  $2T/\pi$  factor, this is an MSK signal, but note that the development in class had a negative sign for the second term which can be put back in place, if necessary, by complementing the odd-numbered data bits . . . .

- (c) Since  $\cos A \cos B \mp \sin A \sin B = \cos(A \pm B)$ , the frequency is  $f_c - (4T)^{-1}$  if  $b_0 = b_1$  and  $f_c + (4T)^{-1}$  if  $b_0 \neq b_1$  which can be summarized as  $f_c - (-1)^{b_0 \oplus b_1} (4T)^{-1}$ .

3. (a) In a DQPSK system, the receiver determines if the current signal point  $\mathcal{Z}_1 = (\mathcal{U}_1, \mathcal{V}_1)$  is closest to
- the previous signal point  $\mathcal{Z}_0 = (\mathcal{U}_0, \mathcal{V}_0)$ : if so, decide 00 was transmitted;
  - the previous signal point  $\mathcal{Z}_0$  rotated by  $\pi/2$ , i.e.,  $j\mathcal{Z}_0 = (\mathcal{V}_0, -\mathcal{U}_0)$ : if so, decide 01;
  - the previous signal point  $\mathcal{Z}_0$  rotated by  $\pi$ , i.e.,  $-\mathcal{Z}_0 = (-\mathcal{U}_0, -\mathcal{V}_0)$ : if so, decide 11;
  - the previous signal point  $\mathcal{Z}_0$  rotated by  $3\pi/2$ , i.e.,  $-j\mathcal{Z}_0 = (-\mathcal{V}_0, \mathcal{U}_0)$ : if so, decide 10.

Finding the smallest (squared) distance is equivalent to finding the largest of  $\pm(\mathcal{U}_0\mathcal{U}_1 + \mathcal{V}_0\mathcal{V}_1)$ , and  $\pm(\mathcal{U}_0\mathcal{V}_1 - \mathcal{V}_0\mathcal{U}_1)$ , that is, looking at the real and imaginary parts of  $\mathcal{Z}_0\mathcal{Z}_1^*$  and

- choosing 00 if  $\Re\{\mathcal{Z}_0\mathcal{Z}_1^*\} > |\Im\{\mathcal{Z}_0\mathcal{Z}_1^*\}|$ ,
- choosing 11 if  $\Re\{\mathcal{Z}_0\mathcal{Z}_1^*\} < -|\Im\{\mathcal{Z}_0\mathcal{Z}_1^*\}|$ ,
- choosing 01 if  $\Im\{\mathcal{Z}_0\mathcal{Z}_1^*\} > |\Re\{\mathcal{Z}_0\mathcal{Z}_1^*\}|$ ,
- choosing 10 if  $\Im\{\mathcal{Z}_0\mathcal{Z}_1^*\} < -|\Re\{\mathcal{Z}_0\mathcal{Z}_1^*\}|$ .

In terms of individual bits, the first bit is 0 or 1 according as  $\Re\{\mathcal{Z}_0\mathcal{Z}_1^*\} + \Im\{\mathcal{Z}_0\mathcal{Z}_1^*\} > 0$  or  $< 0$  while the second bit is 0 or 1 according as  $\Re\{\mathcal{Z}_0\mathcal{Z}_1^*\} - \Im\{\mathcal{Z}_0\mathcal{Z}_1^*\} > 0$  or  $< 0$ .

- (b) The decision rule in a  $\pi/2$ -DPSK system is similar in that the receiver determines whether  $\mathcal{Z}_1$  is closer to  $j\mathcal{Z}_0$  or to  $-j\mathcal{Z}_0$ . Thus, the receiver looks at  $\Im\{\mathcal{Z}_0\mathcal{Z}_1^*\}$  and decides on a 0 or a 1 according as  $\Im\{\mathcal{Z}_0\mathcal{Z}_1^*\} > 0$  or  $< 0$ .