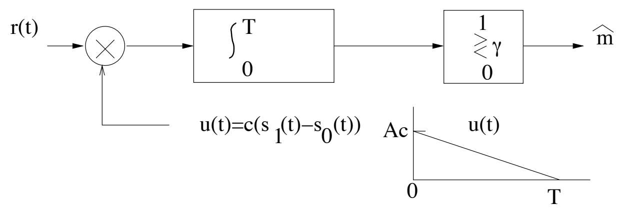
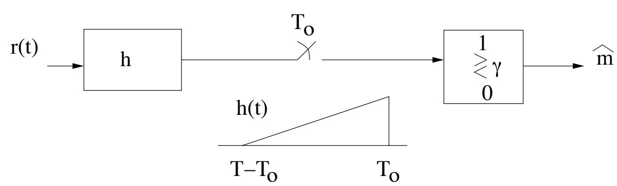
SOLUTIONS TO EXAM I — COMMUNICATIONS II – SPRING 2005

1. Consider a baseband binary communication system with possible transmitted signals $s_1(t) = AI_{[0,T]}(t)$ and $s_0(t) = \frac{At}{T}I_{[0,T]}(t)$. Here $I_{[0,T]}(t)$ is the function that is one if $0 \le t \le T$ and is zero otherwise. Suppose s_1 is sent with probability $\pi_1 = \frac{1}{3}$ and s_0 is sent with probability $\pi_0 = \frac{2}{3}$. The received signal is $r(t) = s_m(t) + n(t)$, where $m \in \{0, 1\}$ and n is white Gaussian noise with two-sided power spectral density $\frac{N_o}{2}$.
(a) Sketch the Bayes optimal receiver in correlator form. Be sure to indicate the value of the threshold in the decision device. (b) Sketch the Bayes optimal receiver using a matched filter and sampler. Be sure to indicate the filter's impulse response function.

(a) Correlate with $u(t) = c(s_1(t) - s_0(t))$, where c is an arbitrary positive constant. Use threshold $\gamma = \frac{m_0+m_1}{2} + \frac{\sigma^2}{m_1-m_0}\ln\frac{\pi_0}{\pi_1} = c\left[\frac{\mathcal{E}_1-\mathcal{E}_0}{2} - \frac{||s_1-s_0||^2 N_o/2}{||s_1-s_0||^2}\ln\frac{\pi_0}{\pi_1}\right] = c\left[\frac{A^2T}{3} + \frac{N_o\ln(2)}{2}\right]$.
(b) Or use the filter $h(t) = u(T_o - t)$ and sample at time T_o.



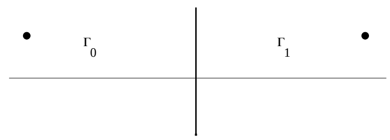


2. Consider a binary communication system with possible transmitted signals s_1 and s_0, where

$$s_m(t) = A\left(a_m\cos(2\pi f_c t + \phi) + \beta\sin(2\pi f_c t + \phi)\right) \qquad 0 \le t \le T,\ m \in \{0, 1\}$$

where $a_1 = 1$ and $a_0 = -1$ and β is a positive constant. As usual, assume that A, T, f_c are positive constants with $f_cT >> 1$, and assume the signal is corrupted by additive white Gaussian noise with 2-sided power spectral density $\frac{N_o}{2}$. The purpose of the sine carrier term is to help the receiver recover the phase ϕ. Suppose that the receiver is able to recover the phase ϕ perfectly, and can therefore correlate the received signal over $[0, T]$ with $\psi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t + \phi)$ to produce Z_1 and with $\psi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t + \phi)$ to produce Z_2.
(a) Sketch the signal coordinates relative to the basis $\{\psi_1, \psi_2\}$, and sketch the maximimum likelihood decision regions. (b) Express the probability of error in terms of A, T, N_o, and the Q function.
(c) What is the cost in dB of using the sine carrier term? In other words, if a perfect phase tracking loop could be implemented without the sine carrier term, what would be the effective difference in required energy per transmitted bit, in dB? (Hint: the answer is not zero.)

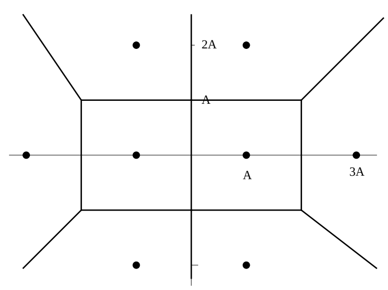
(a) $s_m \leftrightarrow ((s_m, \psi_1), (s_m, \psi_2)) = (a_m \sqrt{\frac{A^2T}{2}}, \beta\sqrt{\frac{A^2T}{2}})$.



(b) $p_e = Q(\sqrt{\frac{||s_1 - s_0||^2}{2N_o}}) = Q(\sqrt{\frac{A^2T}{N_o}})$.
(c) $\mathcal{E}_b = \frac{(1+\beta^2)A^2T}{2}$, so that $p_e = Q(\sqrt{\frac{2\mathcal{E}_b}{(1+\beta^2)N_o}})$. The sine carrier signal thus costs $10\log_{10}(1+\beta^2)$ dB in effective signal energy.

3. Consider an $8-ary$ QAM system with the following signal coordinates relative to an orthonormal basis $\{\psi_1, \psi_2\}$: $(\pm A, 0)$, $(\pm 3A, 0), (\pm A, \pm 2A)$. The signals are sent with equal probability, and are corrupted by AWGN with two-sided power spectral density $\frac{N_o}{2}$.
(a) Sketch the maximum likelihood decision regions on the following signal coordinate diagram.
(b) Express the average symbol energy and the average energy per bit in terms of A^2.
(c) Find a union bound on the probability of symbol error given that the signal with coordinates $(A, 2A)$ is transmitted. Use the minimum number of terms required.
(d) Express the maximum symbol error probability exactly in terms of A, N_o, and the Q function, for the maximum likelihood receiver. (The maximum symbol error probability is the maximum, over all eight possible transmitted symbols, of the symbol error probability.)

(a)



(b) $\mathcal{E}_s = \frac{1}{8}\{2(A^2) + 2(9A^2) + 4(5A^2)\} = 5A^2$ and $\mathcal{E}_b = \frac{\mathcal{E}_s}{3} = \frac{5A^2}{3}$.
(c) The boundary of the decision region for signal $(A, 2A)$ has three line segments, so three terms in the union bound suffice, yielding: $p_{e,(A,2A)} \le 2Q(\sqrt{\frac{2A^2}{N_o}}) + Q(\sqrt{\frac{4A^2}{N_o}})$.
(d) By inspection of the signal coordinate diagram, the maximum symbol error probability is achieved by either of the two signals with coordinates $(\pm A, 0)$. Suppose the signal with coordinates $(A, 0)$ is sent. Then Z_1 and Z_2 are independent, Z_1 is $N(A, \frac{N_o}{2})$, and Z_2 is $N(0, \frac{N_o}{2})$. Thus, the maximum probability of symbol error is given by:
$p_{e,max} = 1 - P[0 \le Z_1 \le 2A \text{ and } -A \le Z_2 \le A] = 1 - P[0 \le Z_1 \le 2A]P[A \le Z_2 \le A] = 1 - (1 - 2Q(\sqrt{\frac{2A^2}{N_o}}))^2$.