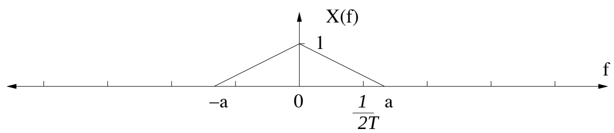
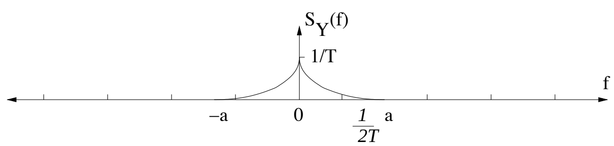
SOLUTIONS TO EXAM II — COMMUNICATIONS II – SPRING 2005

1. Consider the waveform x with the Fourier transform $X(f)$ shown, where $a > 0$. The figure shows the case $a = \frac{1.3}{2T}$, but the relation between the symbol duration T and a can be arbitrary.



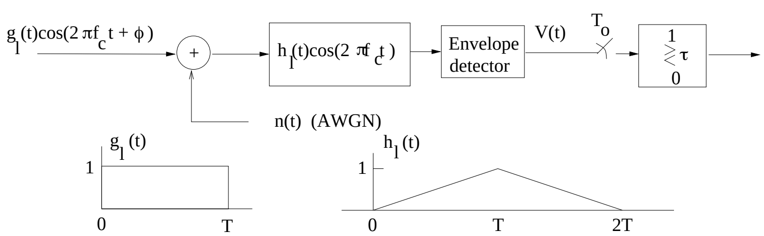
(a) For fixed T, for what value(s) of $a > 0$ is x a Nyquist waveform (not necessarily normalized to have $x(0) = 1$)? (b) Suppose $Y(t) = \sum_{n=-\infty}^{\infty} a_n x(t - nT)$, such that the symbols (a_n) are real valued with $E[a_n a_m] = I_{\{n=m\}}$. Derive and sketch the power spectral density of Y. (c) Determine the one-sided 3dB bandwidth of Y. (d) Of the choices of a which avoid ISI, which is the most bandwidth efficient?

(a) Need $a = \frac{N}{T}$ for some integer $N \geq 1$. (b) $S_Y(f) = \frac{|X(f)^2|}{T} = (1 - |f|/a)^2/T$.



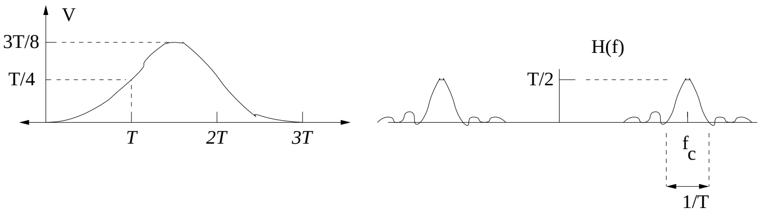
(c) $S_Y(f_{3dB}) = 0.5 S_Y(0)$ yields $(1 - f_{3dB}/a) = \sqrt{0.5}$, or $f_{3dB} = a(1 - \sqrt{0.5}) \approx 0.292a$.
(d) $a = \frac{1}{T}$. (This corresponds to twice the bandwidth of a sinc waveform. Larger values of a are far from bandwidth efficient.)

2. Consider the following noncoherent detector for ON-OFF signaling. The case of an ON signal is shown. Assume that only one bit is sent, so there is no ISI. The ON signal is $g(t) = g_l(t)\cos(2\pi f_c t + \phi)$ and the filter has impulse response $h(t) = h_l(t)\cos(2\pi f_c t)$, where g_l and h_l are pictured. As usual, assume $f_c T >> 1$ and the noise is white with two-sided power spectral density $\frac{N_o}{2}$.



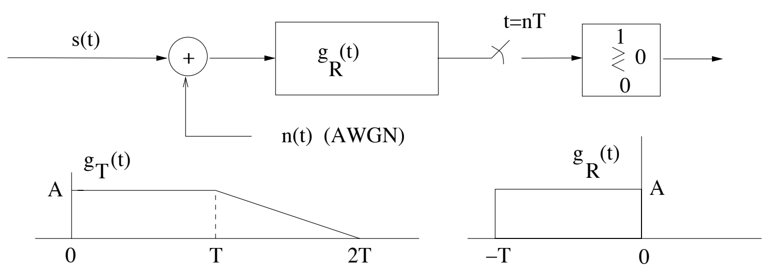
(a) Carefully sketch what the envelope $(V(t))$ would be if there were no noise. (b) Carefully sketch the magnitude of the transfer function of the filter, $H(f)$. (c) Now, taking the noise into account, describe the density of the sample $V(T)$ (In other words, the sampled envelope if $T_o = T$). (d) Suppose the sampling time T_o and threshold τ are adjusted to minimize the average probability of error, assuming ON and OFF signals are equally likely. What is the optimal value of the sampling time, T_o?

(a) Without noise, $V = \frac{1}{2}|g_l * h_l| = \frac{1}{2} g_l * h_l$. (b) See figure.



(c) Without noise, the sample $V(T)$ would equal $\nu = \frac{T}{4}$. The noise out of the filter has power $\sigma^2 = \frac{N_o}{2}(0.5||h_l||^2) = \frac{N_o T}{6}$. Thus, $V(T)$ has the Rician density with parameters ν and σ^2.
(d) The optimal sampling time is $T_o = 3T/2$. This choice leads to the largest value of the parameter ν, making the density f_1 as different as possible from f_0, which is the Rayleigh density with parameter σ^2.

3. Consider a baseband system with transmitted signal $s(t) = \sum_{m=-\infty}^{\infty} a_m g_T(t - mT)$, such that the symbols a_m are independent, with $P[a_m = 1] = P[a_m = -1] = 0.5$ for all m. The signal is subjected to AWGN, but not to channel dispersion. The transmit waveform g_T and receiver filter waveform g_R are pictured. There is intersymbol interference.



(a) What is the peak relative distortion, d^*, if no equalizer is installed? (b) Suppose a three tap equalizer is installed just after the receive filter, with symbol spacing T. What choice of tap weights (c_{-1}, c_0, c_1) minimizes the new peak relative distortion, and what is the resulting new peak relative distortion? (c) Taking into account the Gaussian noise, what is the effective increase in SNR achieved by the equalizer, expressed in dB? In anwering, focus on the maximum bit error probability, with and without the equalizer present. Your answer will be positive if the equalizer yields a reduction in the maximum bit error probability.

(a) The end-to-end waveform $x = g_T * g_R$ satisfies $x(0) = A^2T$, $x(T) = \frac{A^2T}{2}$, and $x(nT) = 0$ for $n \notin \{0, 1\}$. Thus, $d^* = \frac{0.5}{1} = 0.5$.
(b) Letting $q = x * g_E$, we find the nonzero values of $q(nT)$ (dropping the factor A^2T):
$\left|\begin{array}{c|c|c|c} q(-T) & q(0) & q(T) & q(2T) \\ c_{-1} & c_0 + 0.5c_{-1} & c_1 + 0.5c_0 & 0.5c_1 \end{array}\right|$. Setting $q(\pm T) = 0$ yields $\underline{c} = (0, 1, -0.5)$ (or any positive multiple of this). It gives nonzero values $q(0) = 0.$ and $q(2T) = -0.25$, for the new peak relative distortion value 1/4.
(c) The noise samples without the equalizer are independent, mean zero Gaussians. Denote their variance by σ^2. The noise samples out of the equalizer have variance $\sigma^2(1 + (-0.5)^2) = \frac{5\sigma^2}{4}$. Thus, $p_{\max\ before} = Q\left(\sqrt{\frac{2A^2T}{N_o}(1 - 0.5)^2}\right)$ and $p_{\max\ after} = Q\left(\sqrt{\frac{2A^2T}{N_o}\frac{4}{5}(1 - \frac{1}{4})^2}\right) = Q\left(\sqrt{\frac{2A^2T}{N_o}\frac{9}{20}}\right)$ The equalizer thus gives an improvement in worst case error probability, equivalent to a gain of $10\log_{10}(\frac{9}{5})dB \approx 2.55dB$. (Of course no calculators were permitted, and the final numerical answer was not required).