Instructions:

There are **four problems** on this examination.

One page of notes allowed. No other notes, books, tables of integrals, and calculators/personal computers permitted.

Show all your work in the exam booklet provided. Answers without appropriate justification will receive no credit.

Notation: For $\lambda > 0$, $p_\lambda(t) = \begin{cases} 1 & \text{if } 0 \le t < \lambda, \\ 0 & \text{otherwise.} \end{cases}$

$\Phi(x)$ = cumulative probability distribution function for zero-mean unit-variance Gaussian random variable

$Q(x) = 1 - \Phi(x)$

$P_{e,i}$ = probability of error given that signal $s_i(t)$ was transmitted

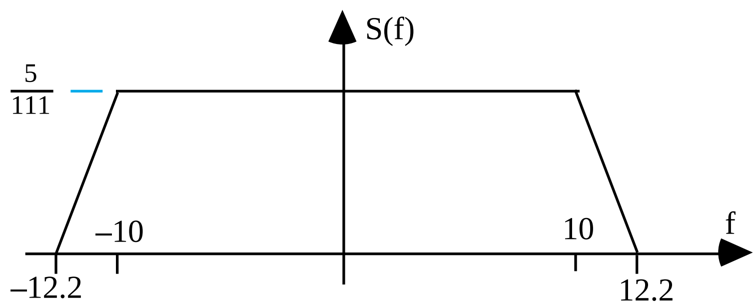
AWGN denotes additive white Gaussian noise with power spectral density $N_0/2$. This random process is independent of the choice of transmitted signal. You may also treat this process as though its mean function is zero.

1. (20 points) Let s(t) denote a Nyquist pulse for symbol interval T seconds, and let S(f) denote its Fourier transform. Then, we know that S(f) satisfies

$$\sum_{n=-\infty}^{\infty} S\left(f+\frac{n}{T}\right) = T, \quad \frac{-1}{2T} \le f \le \frac{1}{2T}$$

Let G(f) denote the sum on the left hand side of the above equation. Then, the equation tells you that G(f) = T for $|f| \le 1/(2T)$. What is G(f) for $|f| > 1/(2T)$? In particular, what is the value of G(1.1/T)?

2. (30 points) A signal s(t) has Fourier transform S(f) as sketched below.



(a) What is s(0)?

(b) If you found that s(0) = 1, find the symbol rates (if any) for which s(t) is a Nyquist pulse.

If you found that s(0) is nonzero but not equal to 1, then find the symbol rates (if any) for which $[s(0)]^{-1}s(t)$ is a Nyquist pulse.

If you found that s(0) = 0, find an explicit formula for s(t). Simply stating that s(t) is the inverse Fourier transform of S(f) or writing s(t) as a Fourier integral is not acceptable.

3. (20 points) Remote controls for consumer electronics (TVs, VCRs, etc) use infrared signals with M-ary orthogonal pulse position modulation (PPM). However, we noted in class that orthogonal code modulation signal sets based on Hadamard matrices have smaller peak power requirements, and simplex signal sets are even more efficient than orthogonal signal sets.

Suppose that you have been assigned the task of designing a remote control for such a consumer electronics application. In 50 or fewer words, explain which of these three signal sets you would use, and why? Keep in mind that because of FCC requirements, you are restricted to using infrared signals; you cannot switch to a different technology.

4. (55 points) Consider a binary baseband digital communication system operating on an AWGN channel. Let τ be a fixed number satisfying $0 \le \tau \le T$. The transmitted signals are

$$s_0(t) = Ap_\tau(t),$$

and $$s_1(t) = A[p_{T/2}(t) - p_{T/2}(t-T/2)].$$

(a) Are $s_0(t)$ and $s_1(t)$ *antipodal* signals for some choice(s) for τ, $0 \le \tau \le T$? Are $s_0(t)$ and $s_1(t)$ *orthogonal* signals for some choice(s) for τ, $0 \le \tau \le T$? If so, specify the value(s) of τ for which the signals $s_0(t)$ and $s_1(t)$ enjoy these properties.

Note: The answers to parts **(b), (d), and (e)** below may differ depending on whether $0 \le \tau \le T/2$ or $T/2 < \tau \le T$, so be sure to consider both cases.

(b) What is the minimax error probability achieved by an *optimum* receiver for this signal set? Note that I am *not* asking you for the details of the matched filter impulse response, the sampling time, the threshold, etc. etc. etc. I just want the minimax error probability of the optimum receiver.

(c) Consider the minimax error probability of part (b) as a *function* of τ, and denote this error probability as $P_e(\tau)$, $0 \le \tau \le T$..

What choice of τ, $0 \le \tau \le T$, minimizes $P_e(\tau)$?

What choice of τ, $0 \le \tau \le T$, maximizes $P_e(\tau)$?

(d) *Now* I am asking you to sketch the impulse response of a matched filter for these signals. Take the sampling instant T_0 to be T. What is the minimax threshold for the matched filter that you found?

(e) Consider a *nonoptimum* receiver whose filter has impulse response $h(t) = p_T(t)$. Find the optimum sampling instant T_0, the minimax threshold γ_m, and the minimax error probability achieved by this nonoptimum filter.

1. $G(f + 1/T) = \sum_{n=-\infty}^{\infty} S\left[\left(f + \frac{1}{T}\right) + \frac{n}{T}\right] = \sum_{n=-\infty}^{\infty} S\left[f + \left(\frac{n+1}{T}\right)\right] = G(f)$, that is, G(f) is periodic in f with period 1/T. Hence, since we know G(f) = T for $|f| \le 1/(2T)$, we get G(f) = T for all f.

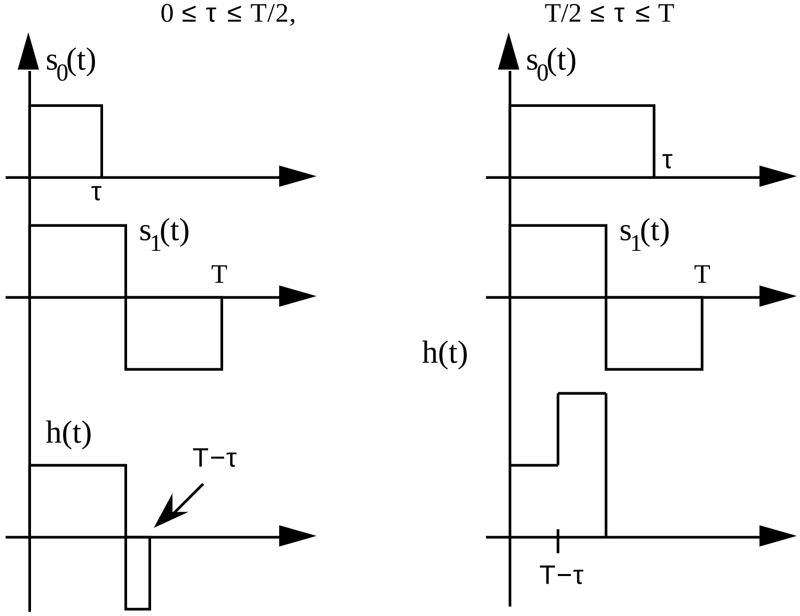
2.(a) $s(0) = \int_{-\infty}^{\infty} S(f)df = \frac{5}{111} \times \frac{20 + 24.4}{2} = \frac{5}{111} \times 22.2 = 1.$

(b) S(f) is the convolution of (1/22.2)rect(f/22.2) and (1/2.2)rect(f/2.2) and hence s(t) = sinc(2.2t)sinc(22.2t). Hence, s(t) is a Nyquist pulse for rates 22.2 symbols/second as well as 2.2 symbols/second.

3. Signal sets based on Hadamard matrices as well as simplex signal sets take on positive as well as negative values (remember that the sum of simplex signals is 0). However, infrared signals are On-Off signals — either a small amount of "heat" is generated at the transmitter or it is not — and hence, orthogonal PPM can be used while Hadamard or simplex signals cannot.

4.(a) The signals are illustrated below for the two cases $0 \le \tau \le T/2$ and $T/2 < \tau \le T$. They are obviously orthogonal if $\tau = 0$ or $\tau = T$. They are not antipodal for any τ.

$0 \le \tau \le T/2$, $T/2 \le \tau \le T$



(b) Almost by inspection (see figure above), $E_0 = A^2\tau$, $E_1 = A^2T$, and

$$s_0, s_1 = \begin{cases} A^2\tau, & \text{if } 0 \le \tau \le T/2, \\ A^2(T-\tau), & \text{if } T/2 \le \tau \le T. \end{cases} \quad \text{Hence, } E_0 + E_1 - 2\, s_0, s_1 = \begin{cases} A^2(T-\tau), & \text{if } 0 \le \tau \le T/2, \\ A^2(3\tau-T), & \text{if } T/2 \le \tau \le T, \end{cases}$$

and the optimum minimax error probability achievable is

$Q\left(\sqrt{\frac{A^2(T-\tau)}{2N_0}}\right)$ if $0 \le \tau \le T/2$, and $Q\left(\sqrt{\frac{A^2(3\tau-T)}{2N_0}}\right)$ if $T/2 \le \tau \le T$.

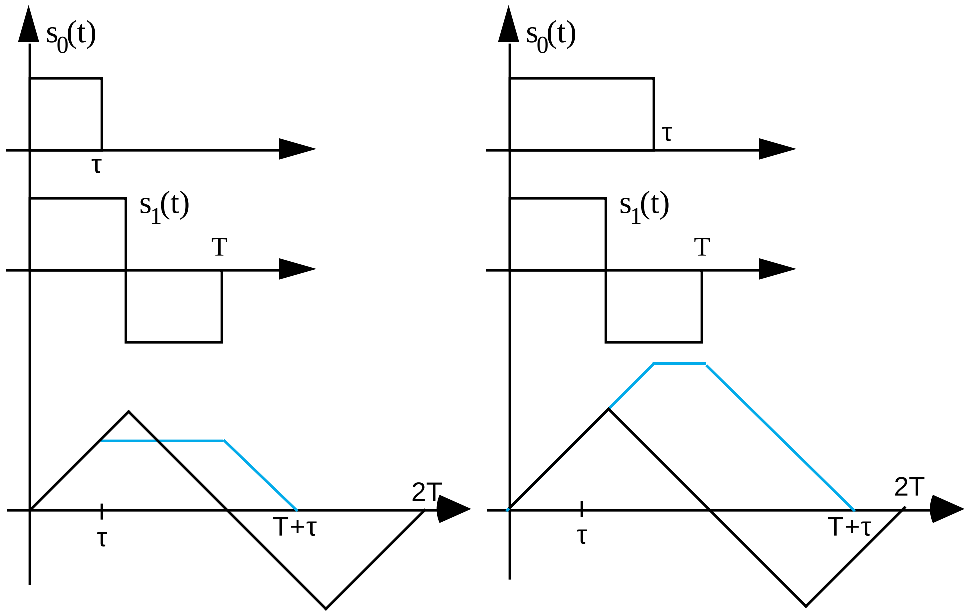
(c) $E_0 + E_1 - 2\, s_0, s_1$ decreases linearly from A^2T at $\tau = 0$ to $A^2T/2$ at $\tau = T/2$, and then increases linearly to $2A^2T$ at $\tau = T$. Hence, the least value of the minimax error probability occurs when $\tau = T$. The signals are orthogonal (with $E_0 + E_1 = A^2T$) in this case. The signals are also orthogonal if $\tau = 0$ but $E_0 + E_1$ is only $A^2T/2$ in this case. Thus, $\tau = 0$ gives a local minimum but not the global minimum for the minimax error probability. Similarly, the maximum error probability occurs if $\tau = T/2$. Note that these results cannot be derived by the usual method of finding where the derivative is zero (why not?), but are obvious if you sketch $(SNR)^2$ as a function of τ.

(d) I take $h(t) = [s_0(T-t) - s_1(T-t)]$. The impulse response functions are as shown above. For this choice of matched filter response, the minimax threshold can be expressed as $(E_0 - E_1)/2$. Thus, we have

$$\gamma_m = (E_0 - E_1)/2 = \begin{cases} -A^2(T-\tau)/2, & \text{if } 0 \le \tau \le T/2, \\ +A^2(T-\tau)/2, & \text{if } T/2 \le \tau \le T. \end{cases}$$

(e) The filter outputs are as sketched below.

$0 \le \tau \le T/2$, $\qquad\qquad$ $T/2 \le \tau \le T$



If $\mathbf{0 \le \tau \le T/2}$, then $T_0 = 3T/2$ at which time, $\hat{s}_0(T_0) = 0$, $\hat{s}_1(T_0) = -AT/2$ and the minimax threshold is $-AT/4$. Notice that this will also hold if $\tau = 0$ (in which case $\hat{s}_0(t) = 0$ for all t.) Since the noise variance is $\sigma^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} h^2(t)dt = \frac{N_0 T}{2}$, the minimax error probability is $Q\left(\frac{\hat{s}_0(T_0) - \hat{s}_1(T_0)}{2\sigma}\right) = Q\left(\sqrt{\frac{AT/4}{N_0 T/2}}\right)$ $= Q\left(\sqrt{\frac{A^2 T}{8N_0}}\right)$ independent of the value of τ. In contrast, the optimum receiver has $P_e = Q\left(\sqrt{\frac{A^2 T}{2N_0}}\right)$ at $\tau = 0$ increasing to $Q\left(\sqrt{\frac{A^2 T}{4N_0}}\right)$ at $\tau = T/2$.

Note that at $\tau = 0$, the optimum receiver has a 6 dB advantage over the (simpler?) nonoptimum receiver (in terms of the value of A required to achieve a given error probability). The improvement decreases to a 3dB advantage at $\tau = T/2$. The optimum receiver can also use a sampling time of $T_0 = T$ whereas the nonoptimum receiver uses a sampling time of 3T/2 (i.e. has a greater delay).

If $\mathbf{T/2 \le \tau \le T}$, any T_0, $T \le T_0 \le 3T/2$, is an optimum sampling time!
Choosing $T_0 = T$, $\hat{s}_0(T_0) = A\tau$, $\hat{s}_1(T_0) = 0$, and the minimax threshold is $A\tau/2$.

The minimax error probability is $Q\left(\frac{A\tau}{2\sqrt{N_0 T/2}}\right) = Q\left(\sqrt{\frac{A^2\tau^2}{2N_0 T}}\right)$

This has value $Q\left(\sqrt{\frac{A^2 T}{8N_0}}\right)$ at $\tau = T/2$ decreasing to $Q\left(\sqrt{\frac{A^2 T}{2N_0}}\right)$ at $\tau = T$. In comparison, the optimum receiver achieves $Q\left(\sqrt{\frac{A^2 T}{4N_0}}\right)$ at $\tau = T/2$ decreasing to $Q\left(\sqrt{\frac{A^2 T}{N_0}}\right)$ at $\tau = T$. Thus, the optimum receiver has a 3dB advantage over the nonoptimum receiver for all τ, $T/2 \le \tau \le T$.